Finite temperature QCD with chiral (overlap) fermions

A. Yu. Kotov
in collaboration with **Sz. Borsanyi, T. Kovacs, K. Szabo, Z. Fodor**

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PoS LATTICE2023 (2024) 179, arXiv: 2401.02750
+ new results

JÜLICH Forschungszentrum

New Trends in Thermal Phases of QCD,
Bratislava, 2026

Outline

- Lattice setup
- Topological susceptibility χ
- Nature of the chiral phase transition
- Spectrum of the Dirac operator and localisation

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- Keep $Q = \mathrm{const}$ ($Q = 0$)
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- $N_t = 8,10$
- Aspect ratios: $N_s/N_t = 3,4,5$
- Everything is preliminary!

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Topological susceptibility from simulations at fixed Q

Correlator method

$$\lim_{|z|\to \text{large}} \langle q(z)q(0)\rangle = \frac{1}{V}\left(Q^2/V - \chi\right) + O(V^{-2}) + O(e^{-m_{\eta'}z})$$

$q(z)$: topological charge density

[Aoki, Fukaya, Hashimoto, Onogi, 2007]

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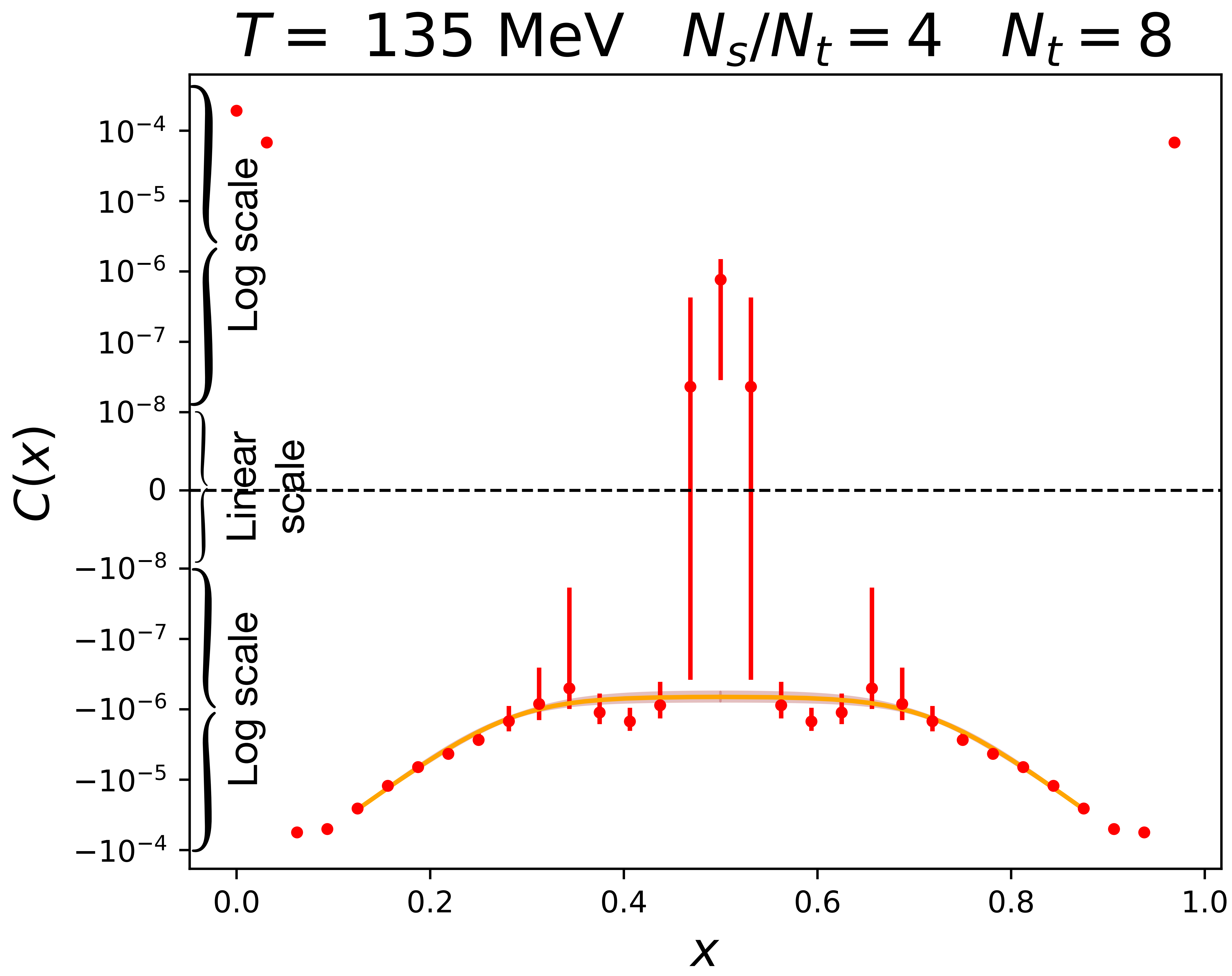
- More stable for $N_s/N_t \gtrsim 4$

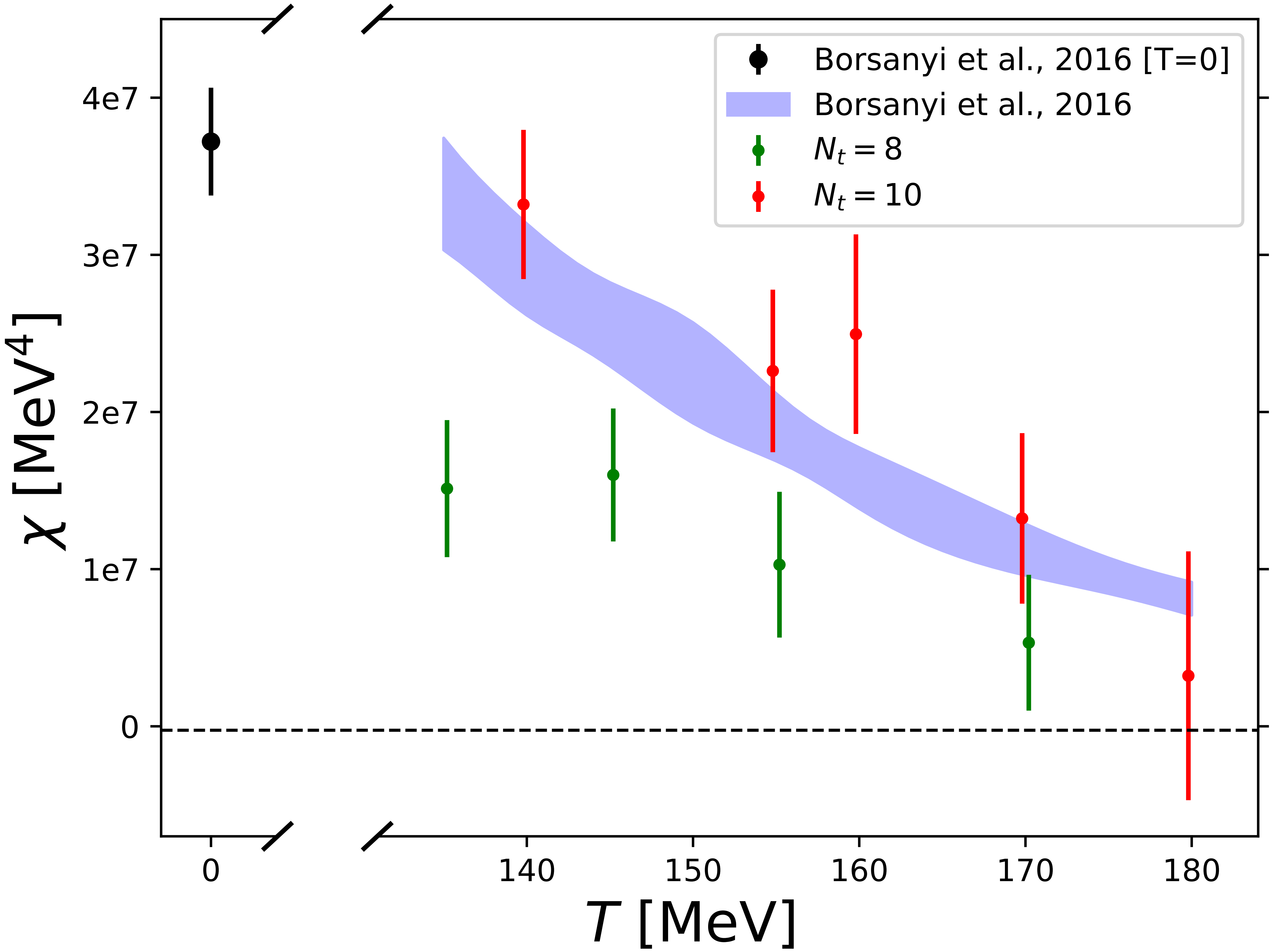


Topological susceptibility from simulations at fixed Q

Correlator method

- Noisy
- Consistent with

  [Borsanyi et al., 2016]
- Local topological fluctuations
- $N_t = 8$: lattice artefacts?



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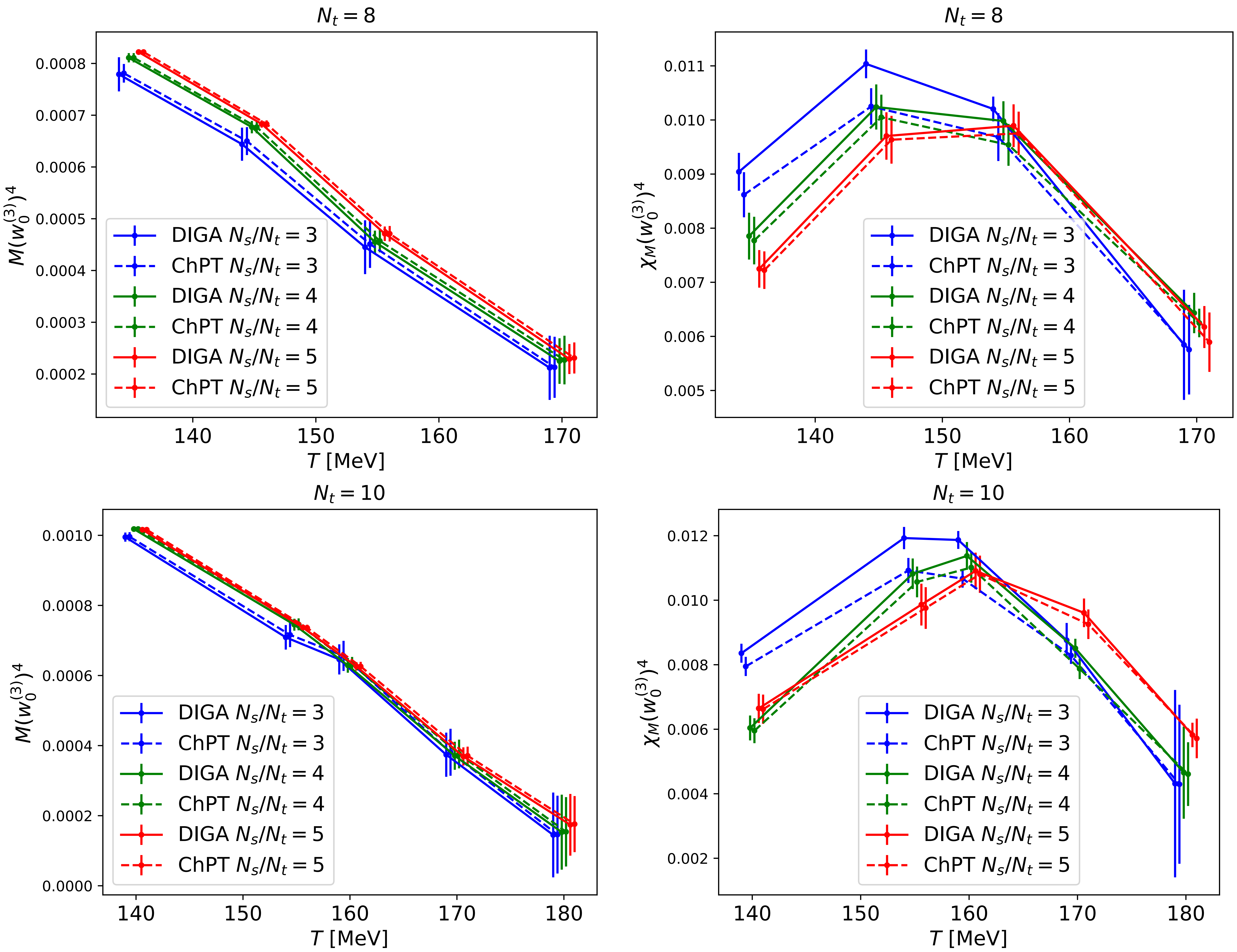
- Summation over sectors: $\langle O \rangle \to \sum_Q \frac{Z_Q}{Z} \langle O \rangle_Q$
- Weights of different sectors:
  - LO ChPT: $Z_Q \sim \int d\theta \frac{I_1(a\cos(\theta/2))}{a\cos(\theta/2)} e^{iQ\theta}$ [Leutwyler, Smilga, 1992]
  - DIGA: $Z_Q \sim I_Q(\chi V)$

Chiral phase transition



Dirac operator spectrum, $T = 145\ \mathrm{MeV}$

$$D_{\mathrm{ov}}^{\dagger} D_{\mathrm{ov}} | e_i \rangle = \lambda_i^2 | e_i \rangle$$

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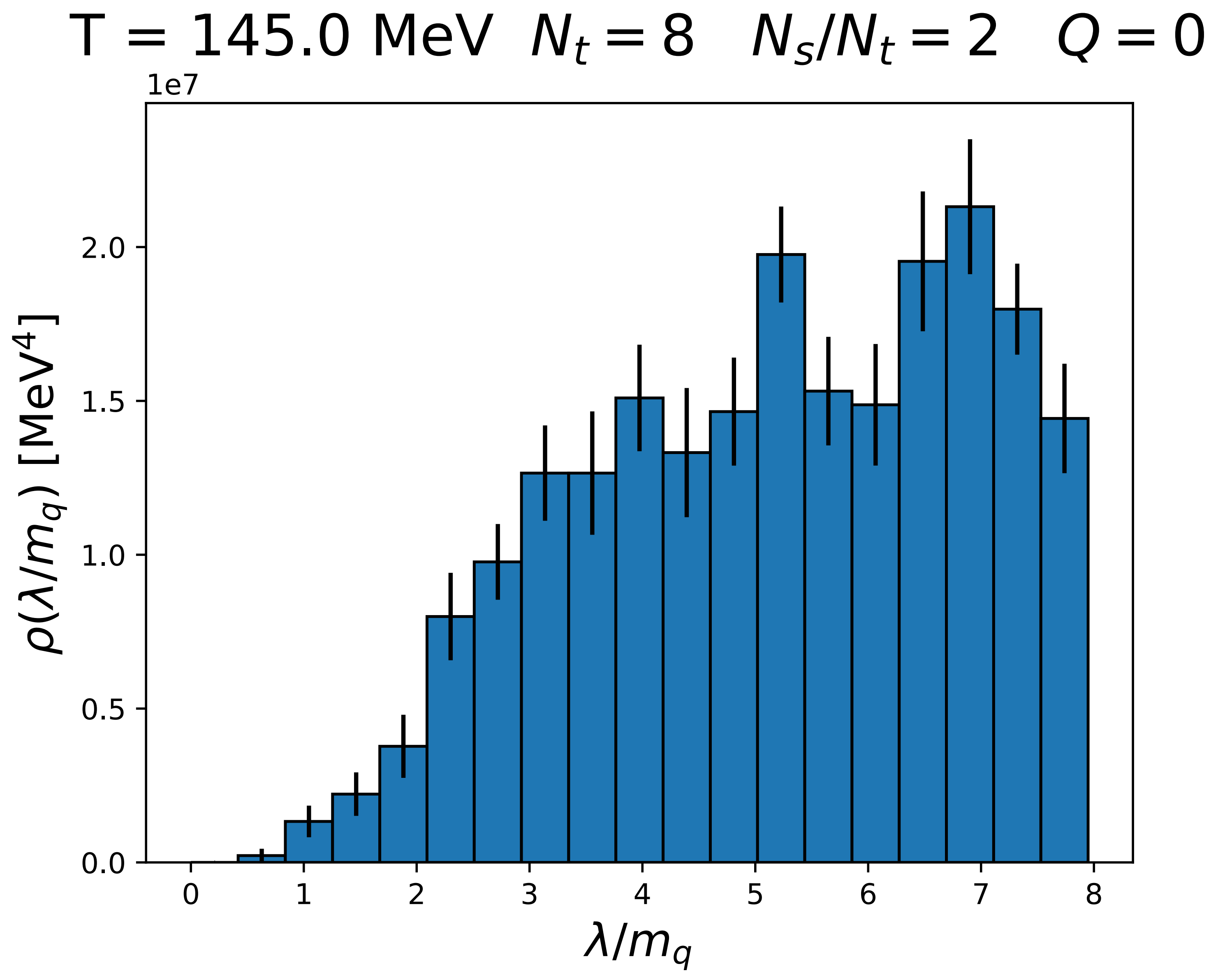

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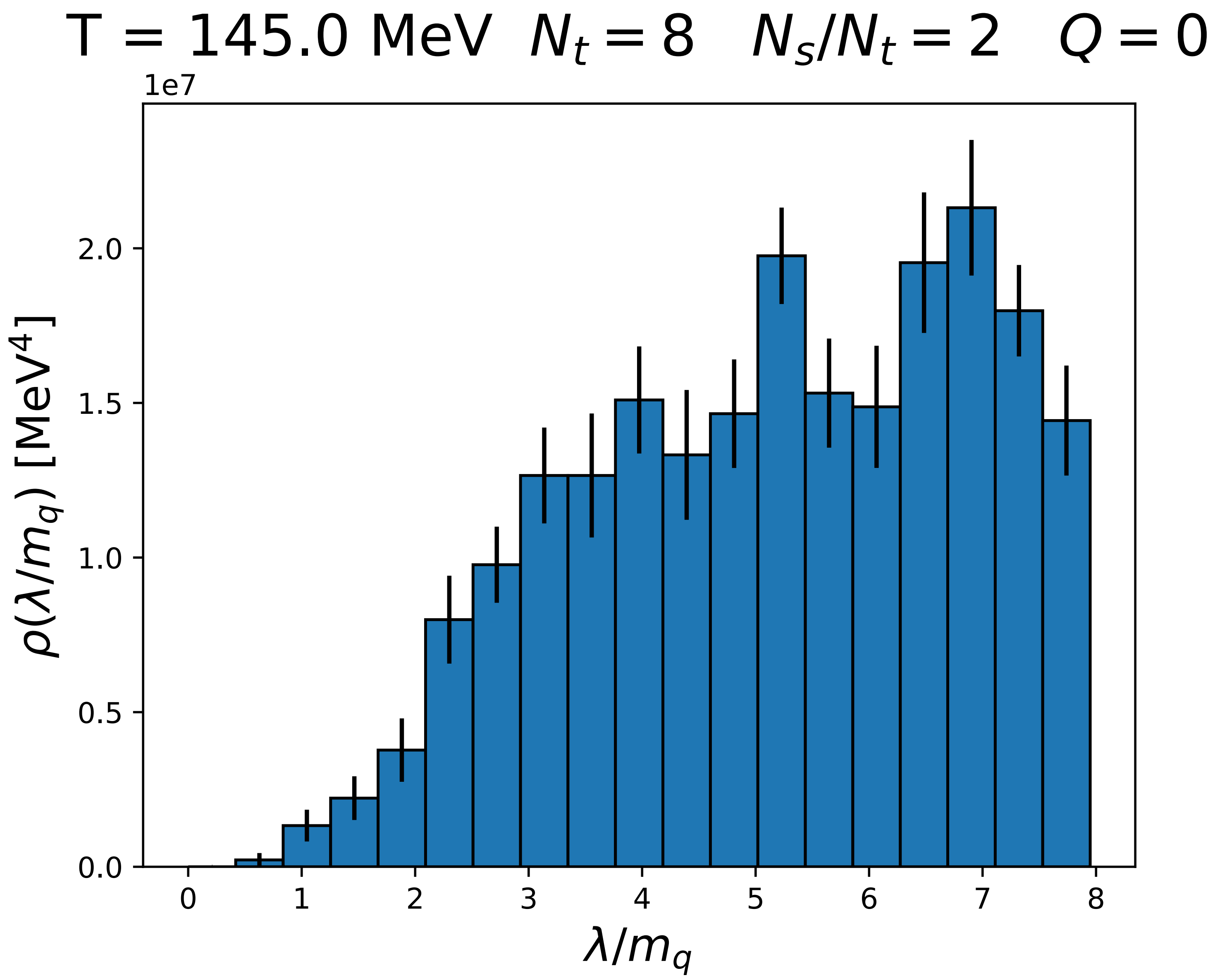

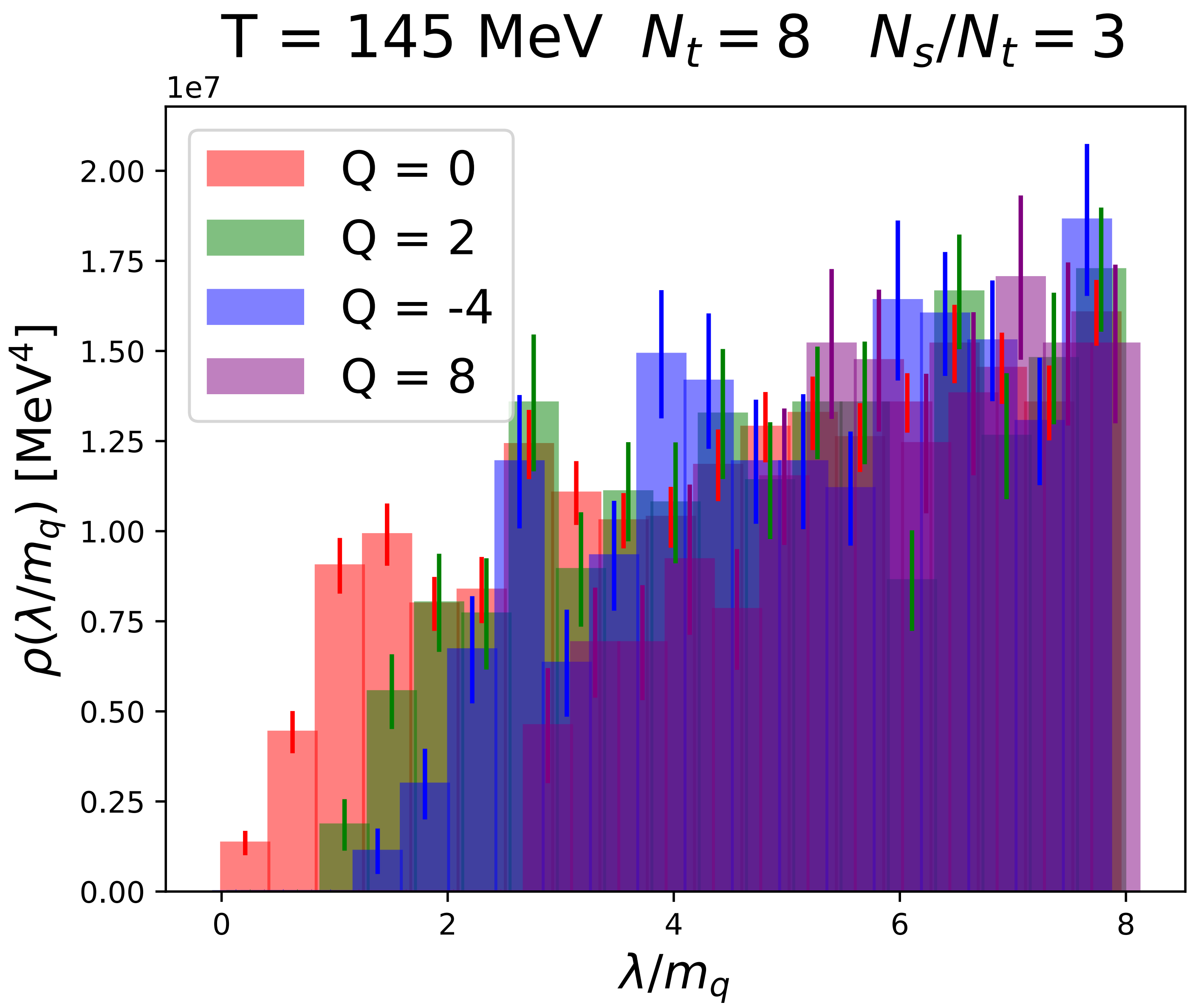

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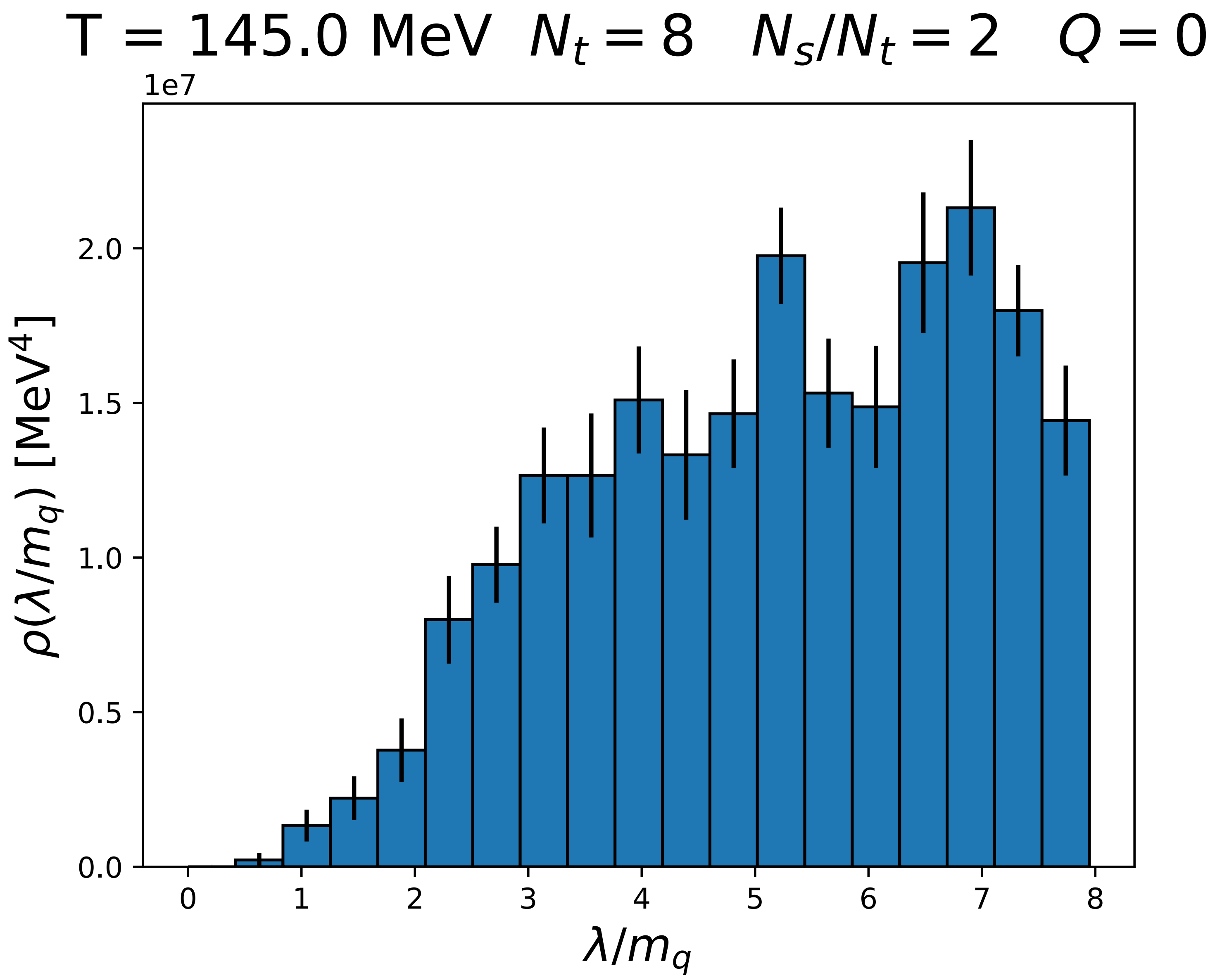

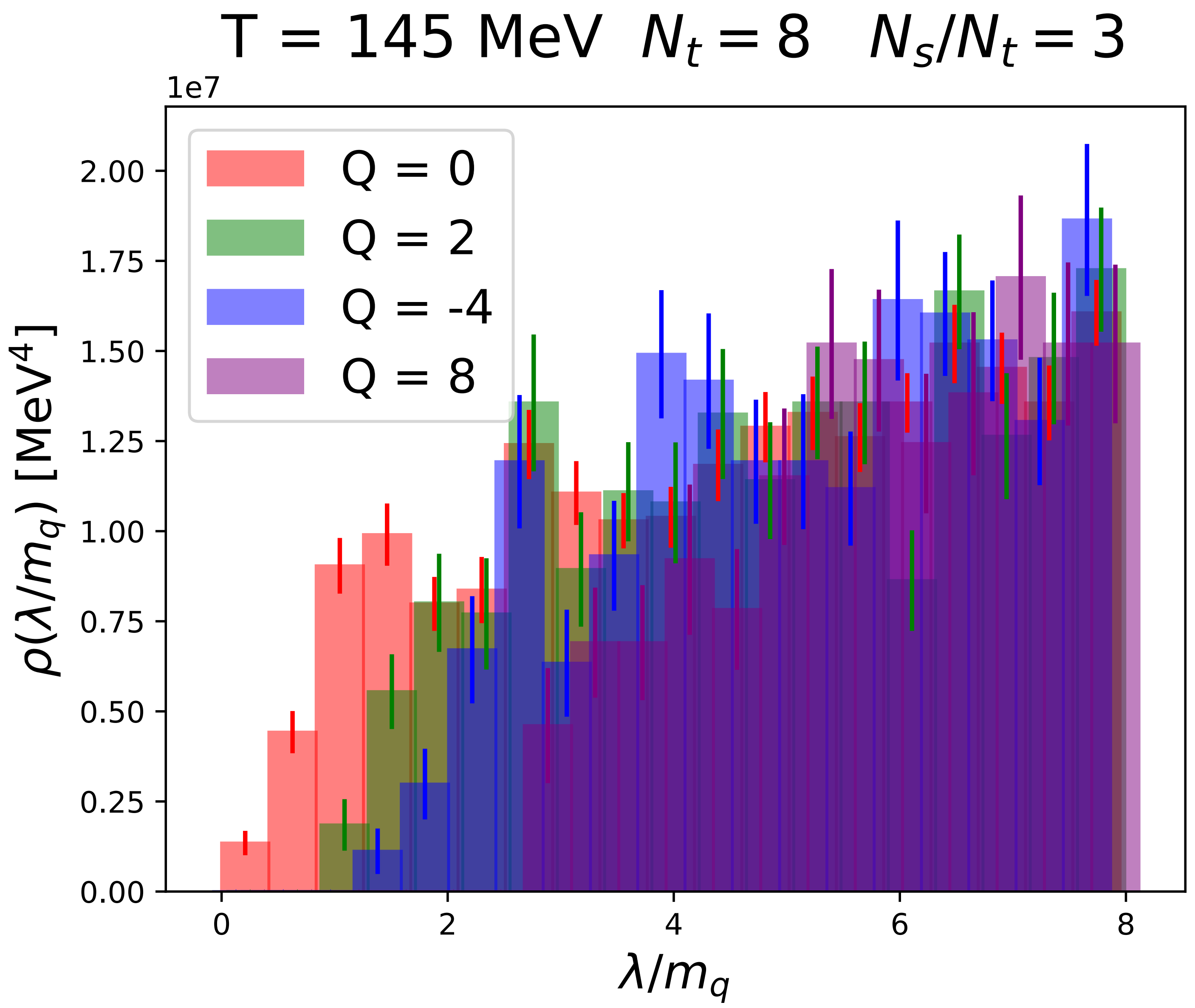

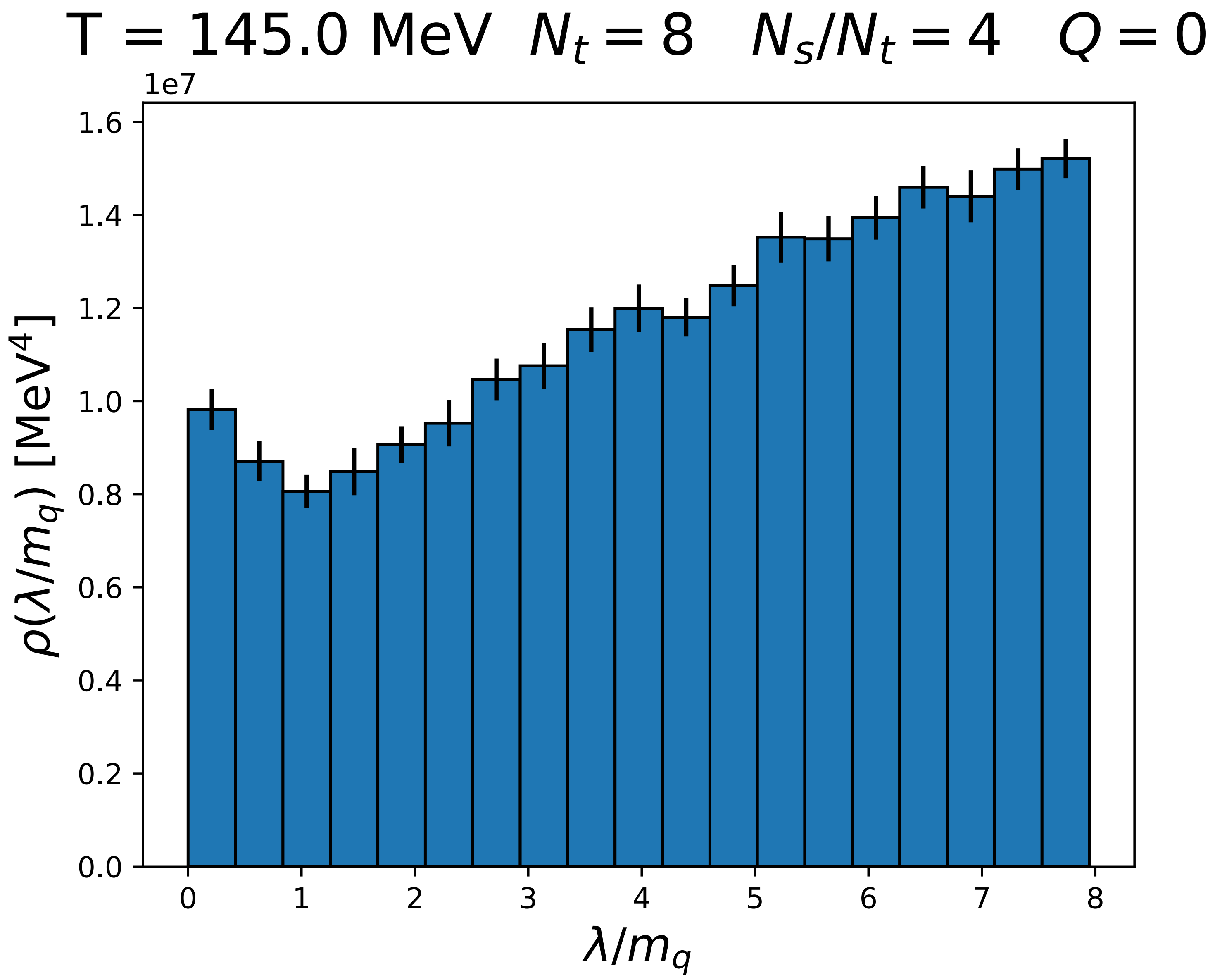

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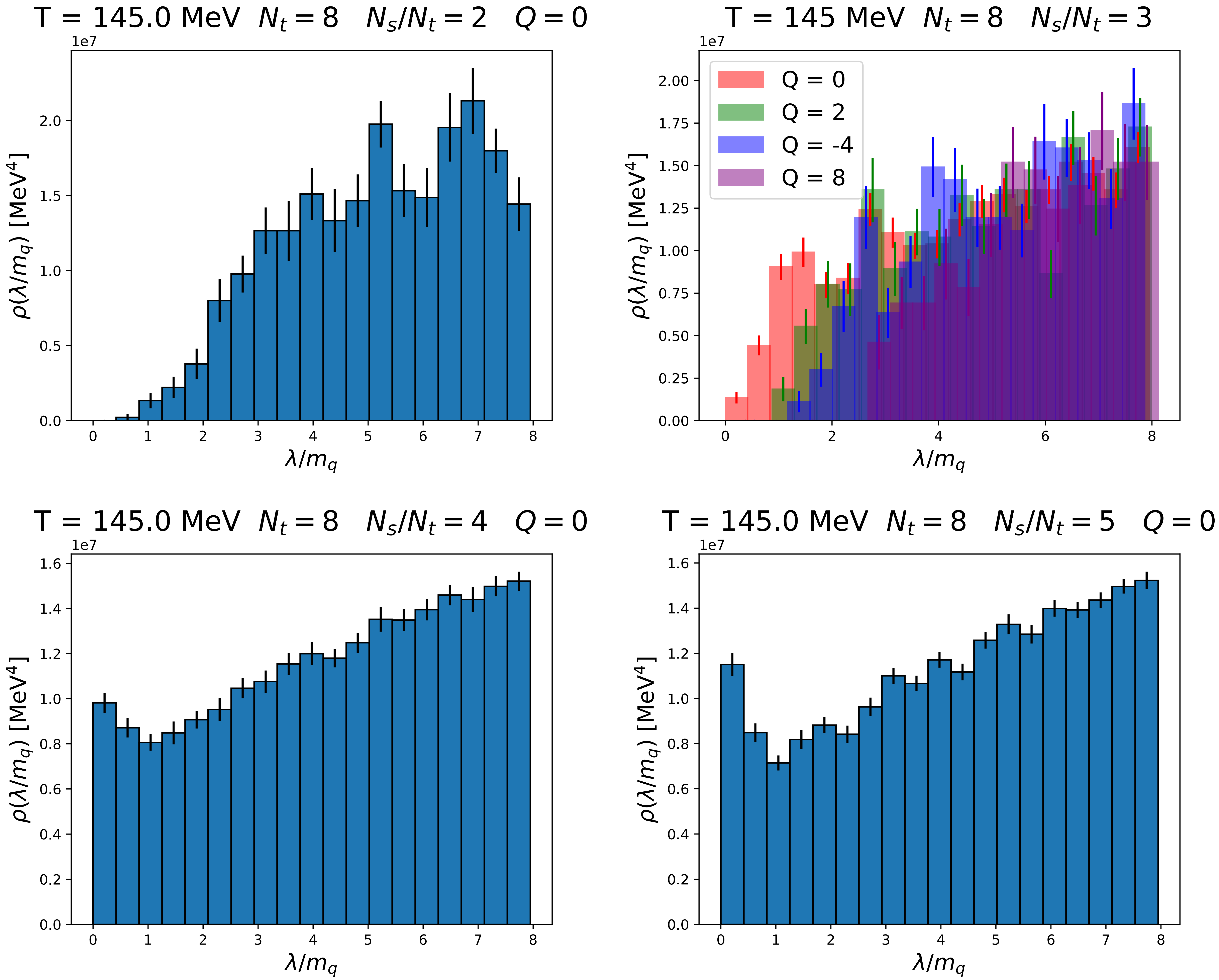
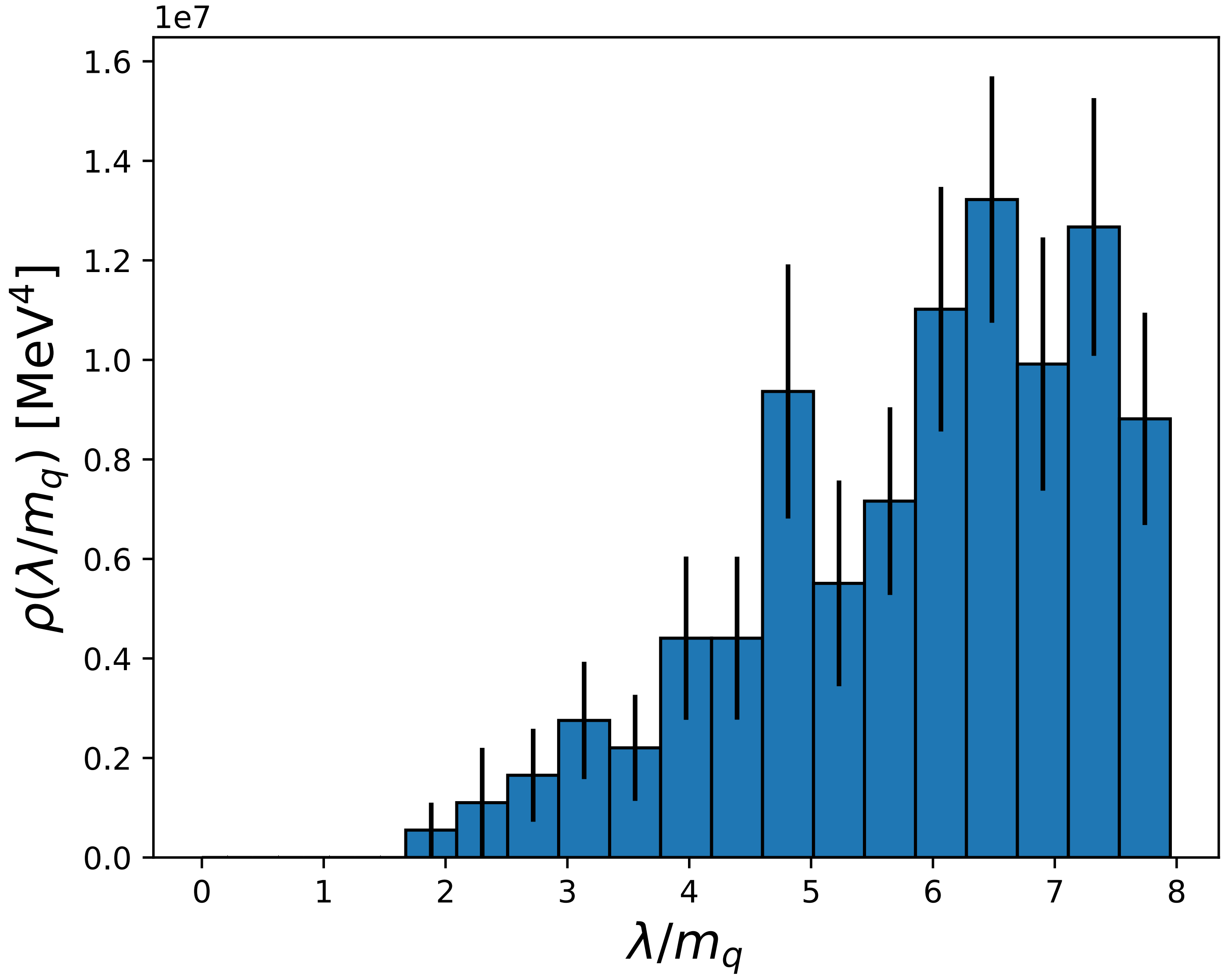


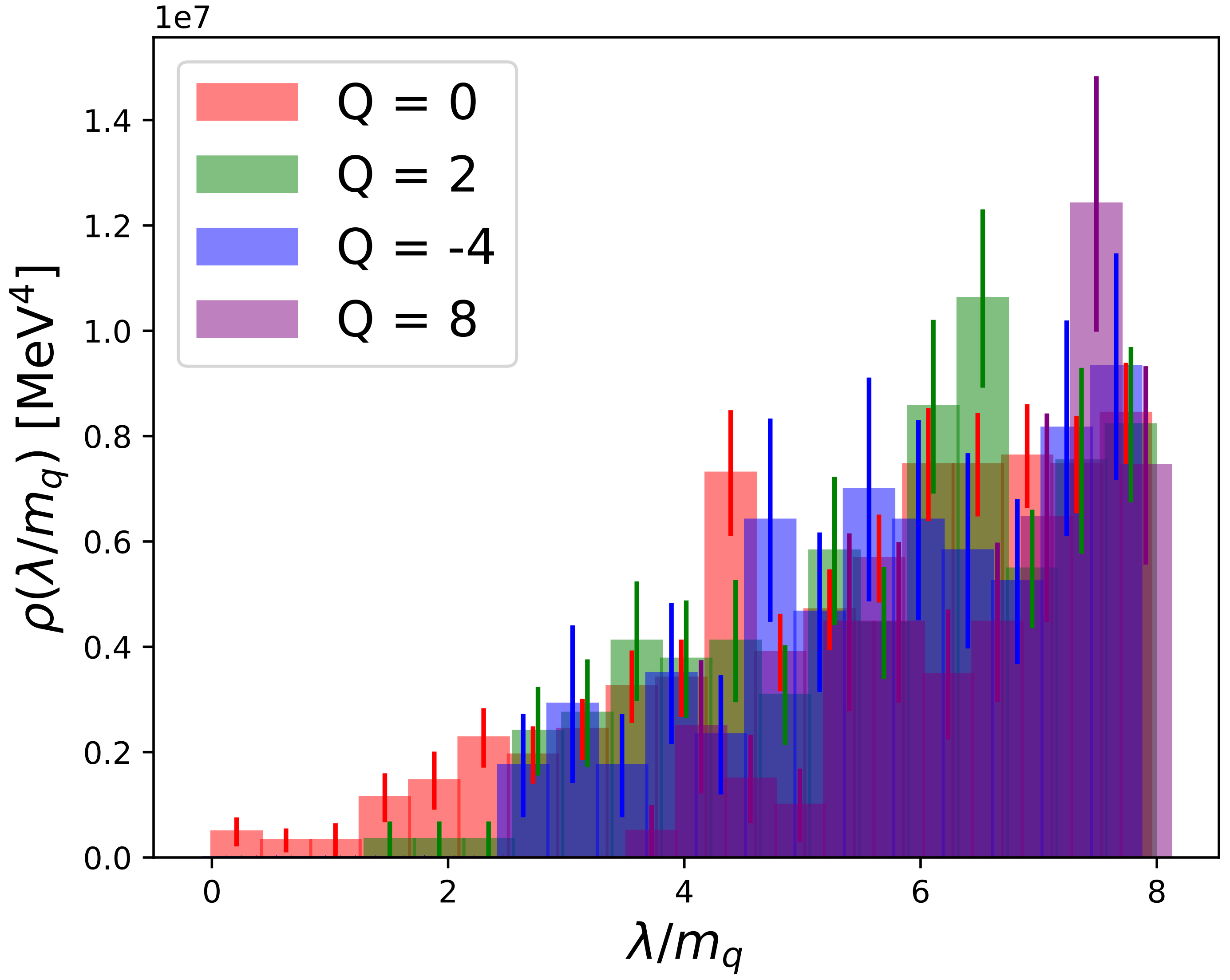
Dirac operator spectrum, $T = 170\ \mathrm{MeV}$

$$D_{\mathrm{ov}}^{\dagger} D_{\mathrm{ov}} \,|\, e_i \rangle = \lambda_i^2 \,|\, e_i \rangle$$

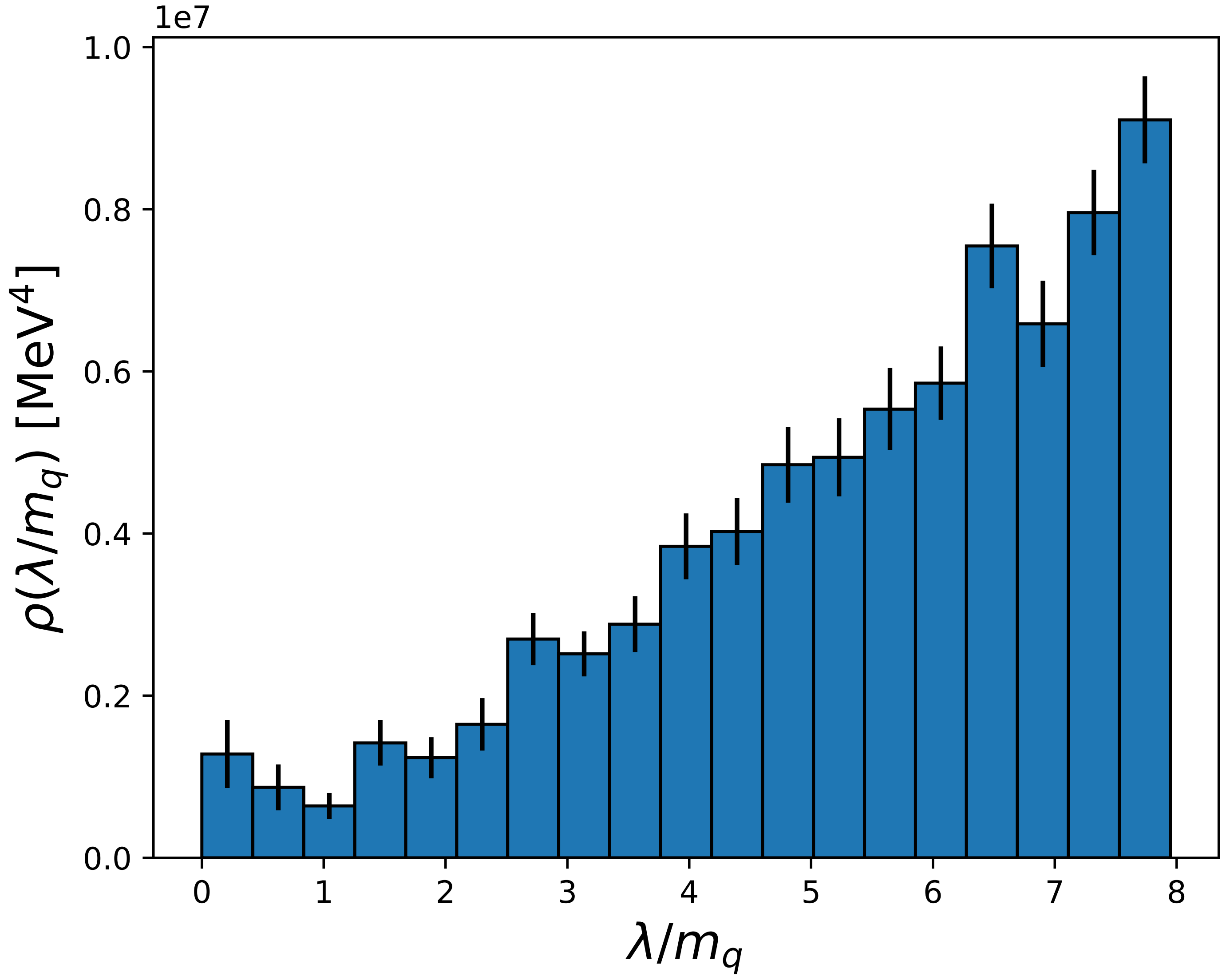
T = 170.0 MeV $N_t = 8$ $N_s/N_t = 2$ $Q = 0$



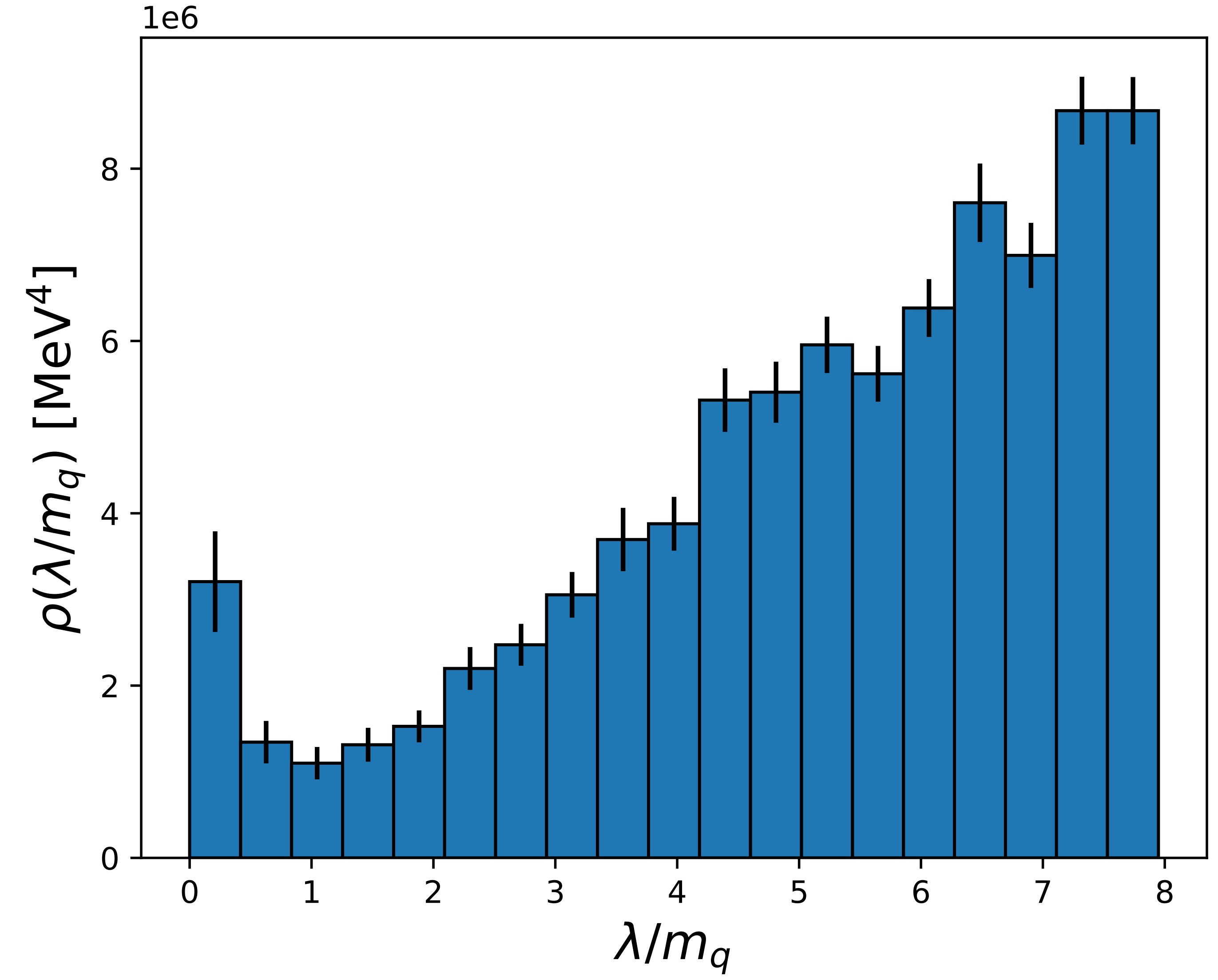
T = 170 MeV $N_t = 8$ $N_s/N_t = 3$



T = 170.0 MeV $N_t = 8$ $N_s/N_t = 4$ $Q = 0$



T = 170.0 MeV $N_t = 8$ $N_s/N_t = 5$ $Q = 0$



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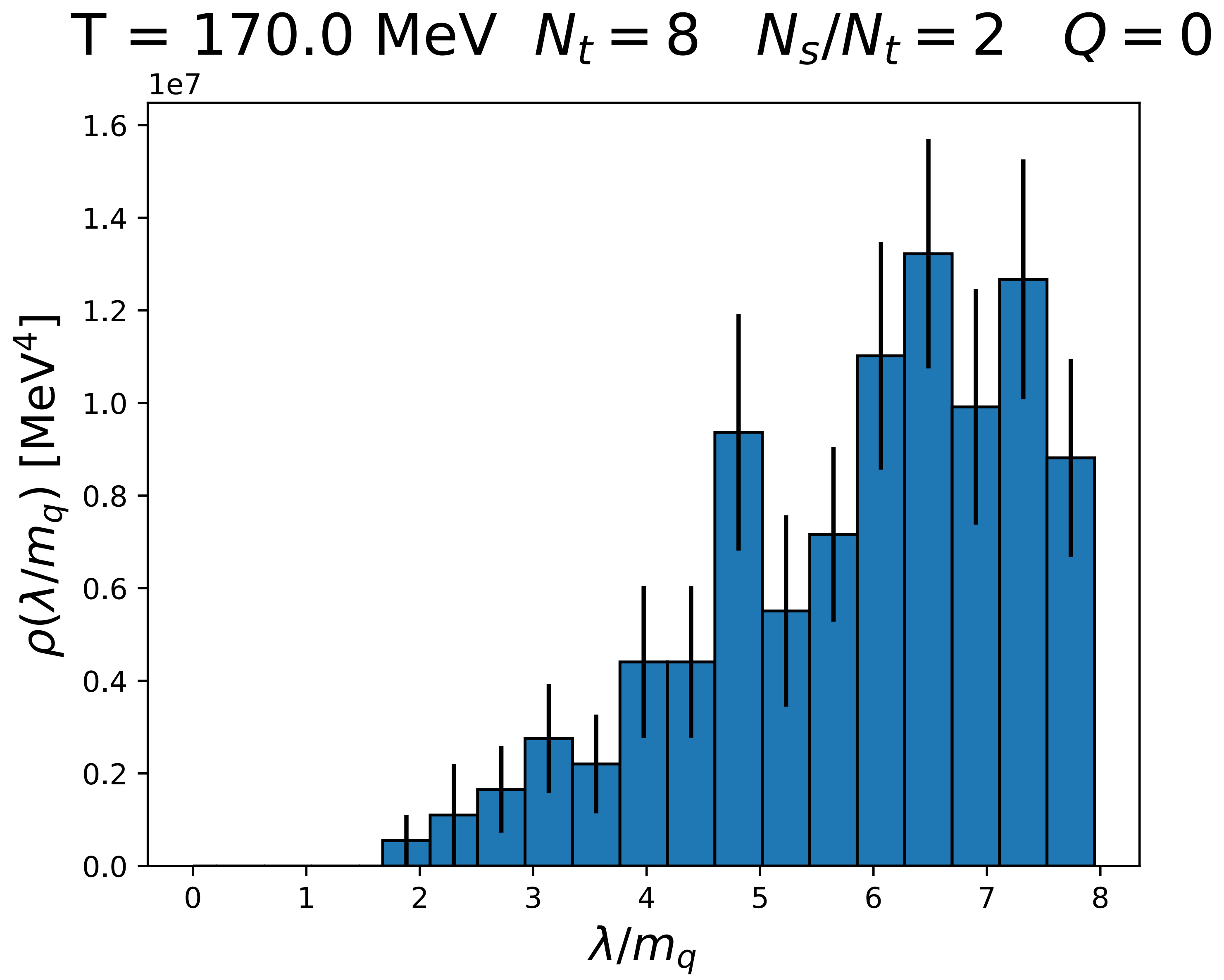
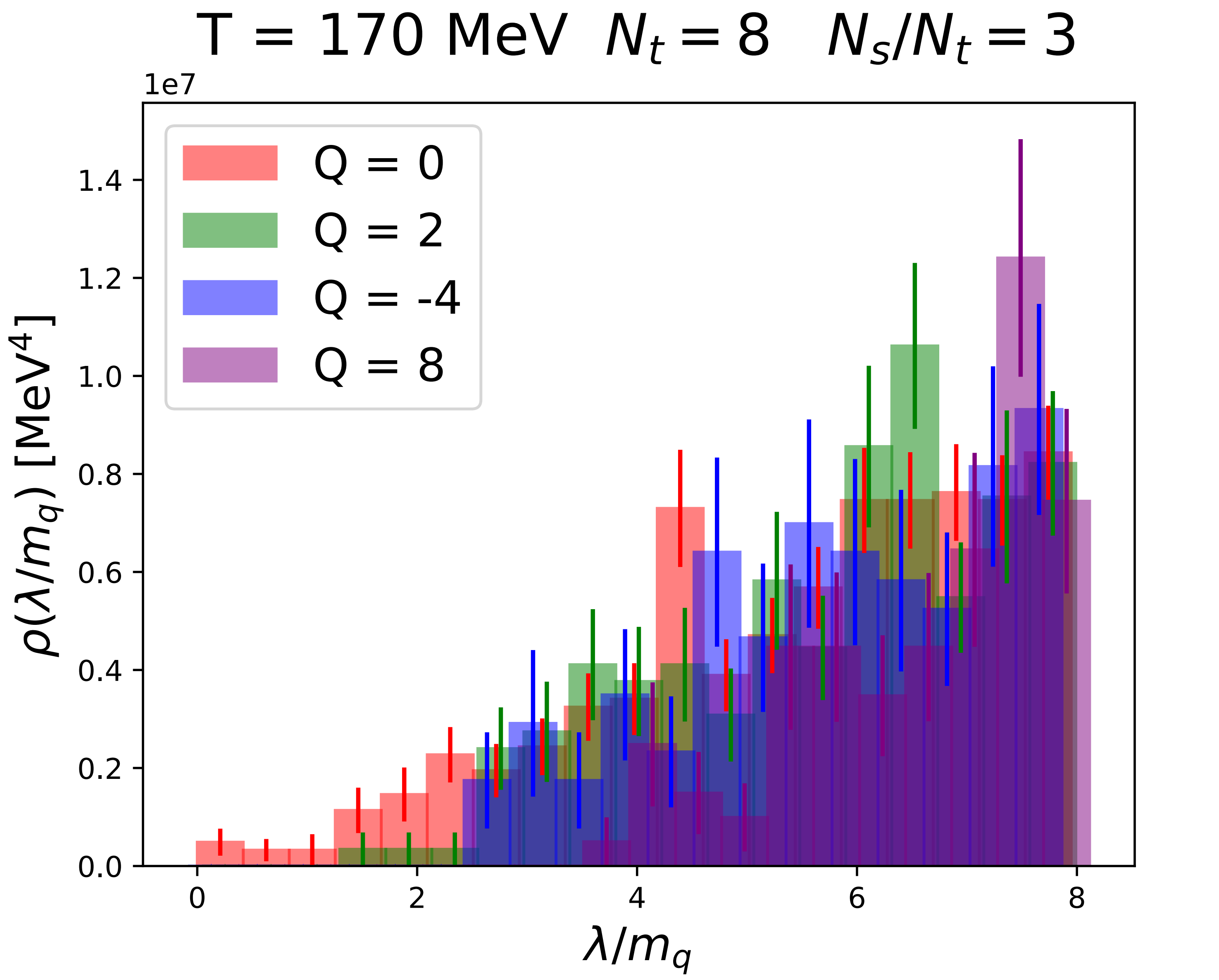
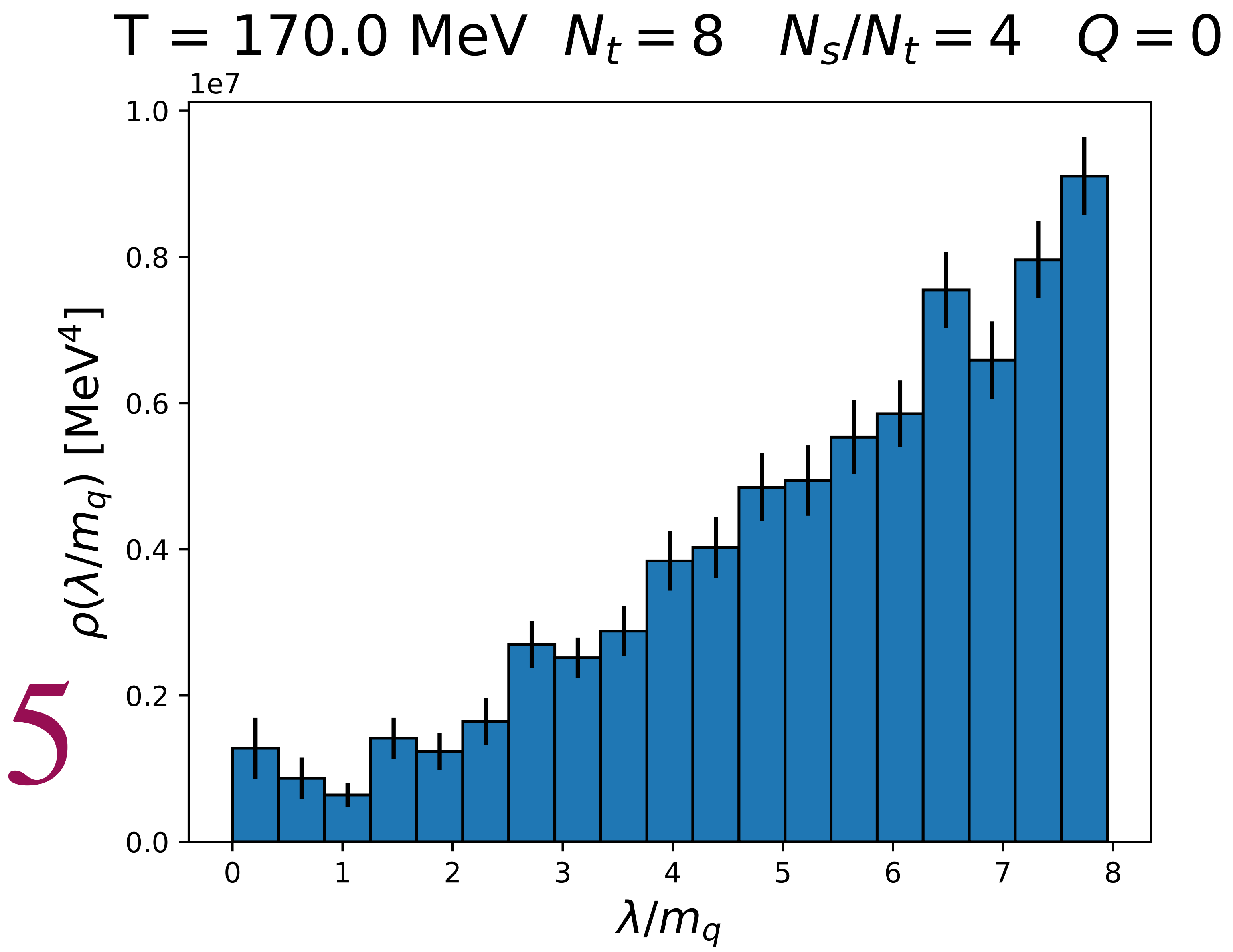
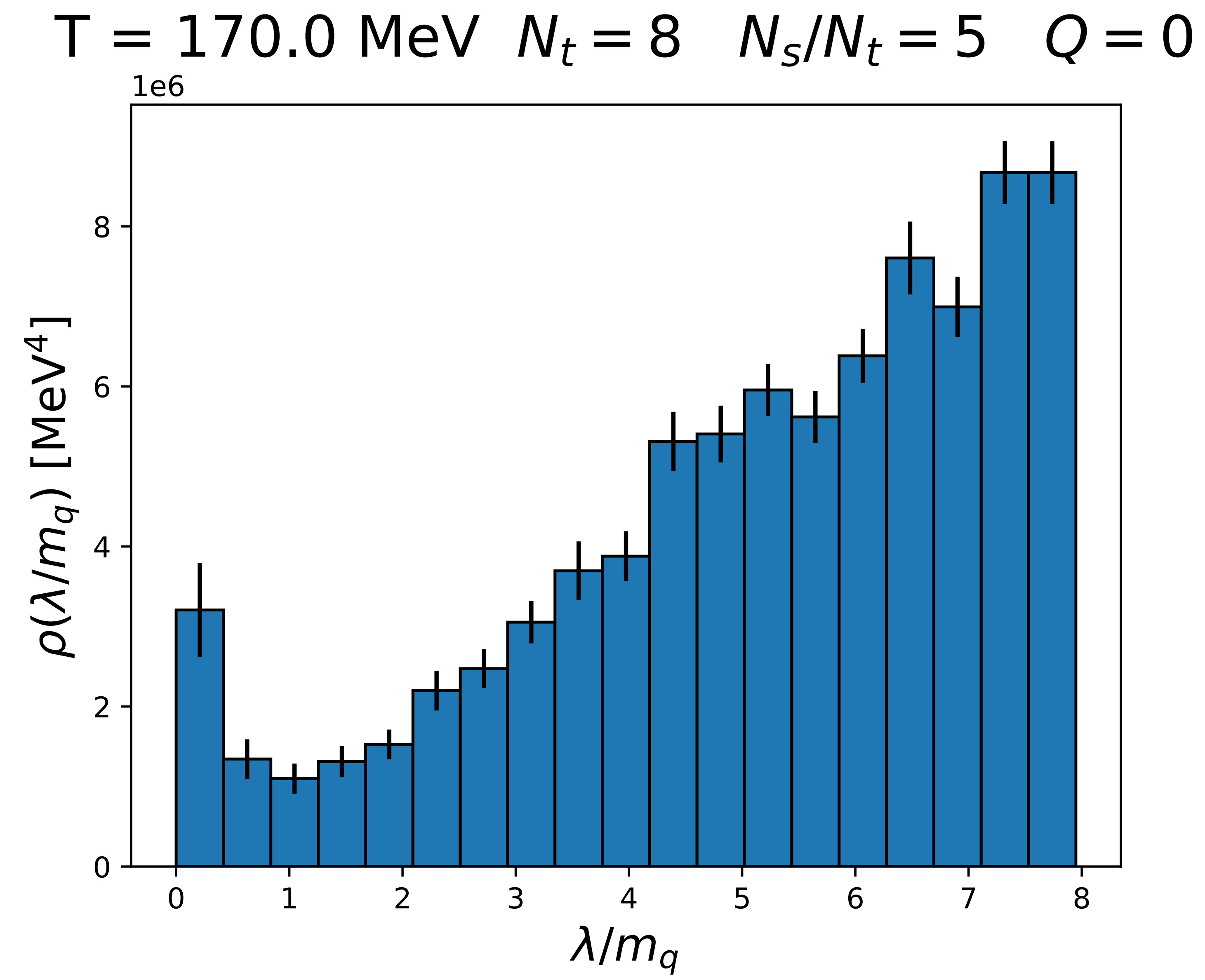








Peak $\rho(\lambda \to 0)$:
Large $N_s/N_t \gtrsim 4 - 5$

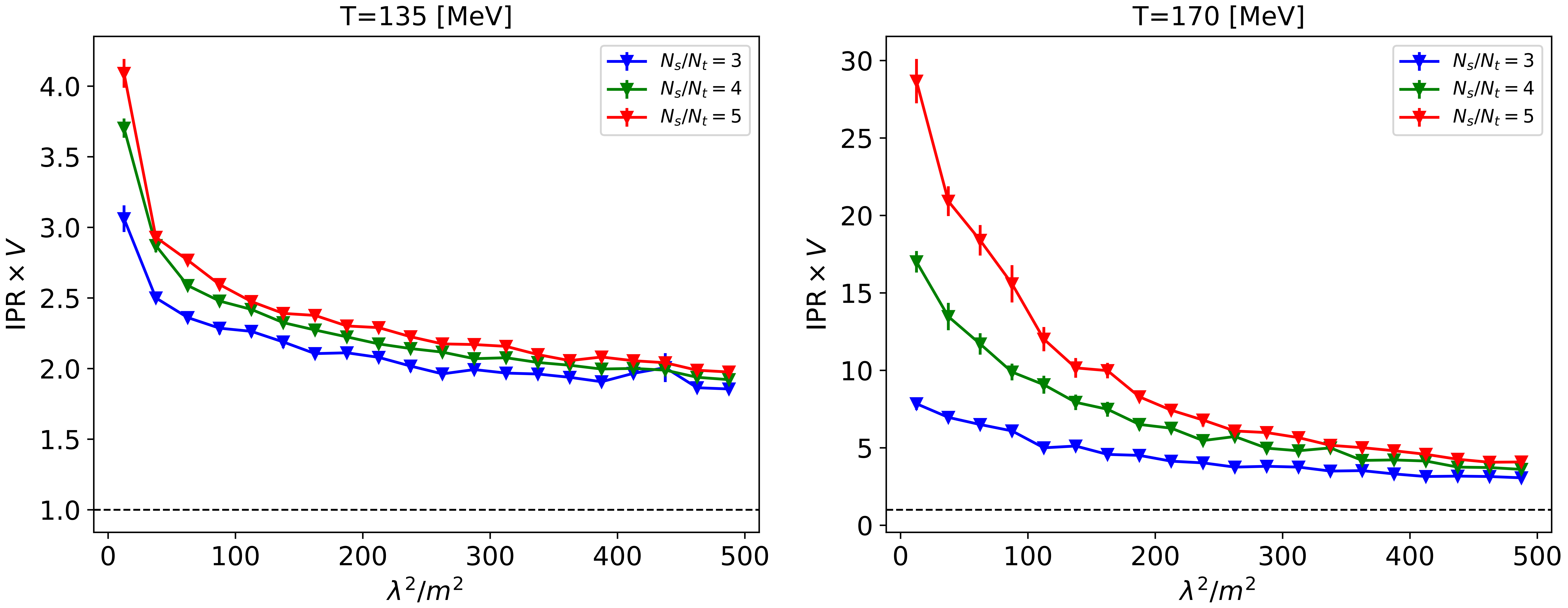
Localisation

Eigenmodes of D

- $\mathrm{IPR}_n = \sum_x ||\psi_n(x)||^4$
- $\mathrm{IPR}(\lambda) = \dfrac{\langle \sum_n \delta(\lambda - \lambda_n)\mathrm{IPR}_n \rangle}{\langle \sum_n \delta(\lambda - \lambda_n) \rangle}$
    - Localised: $\mathrm{IPR} \times V \sim V$
    - Delocalised: $\mathrm{IPR} \times V \sim \mathrm{const}$

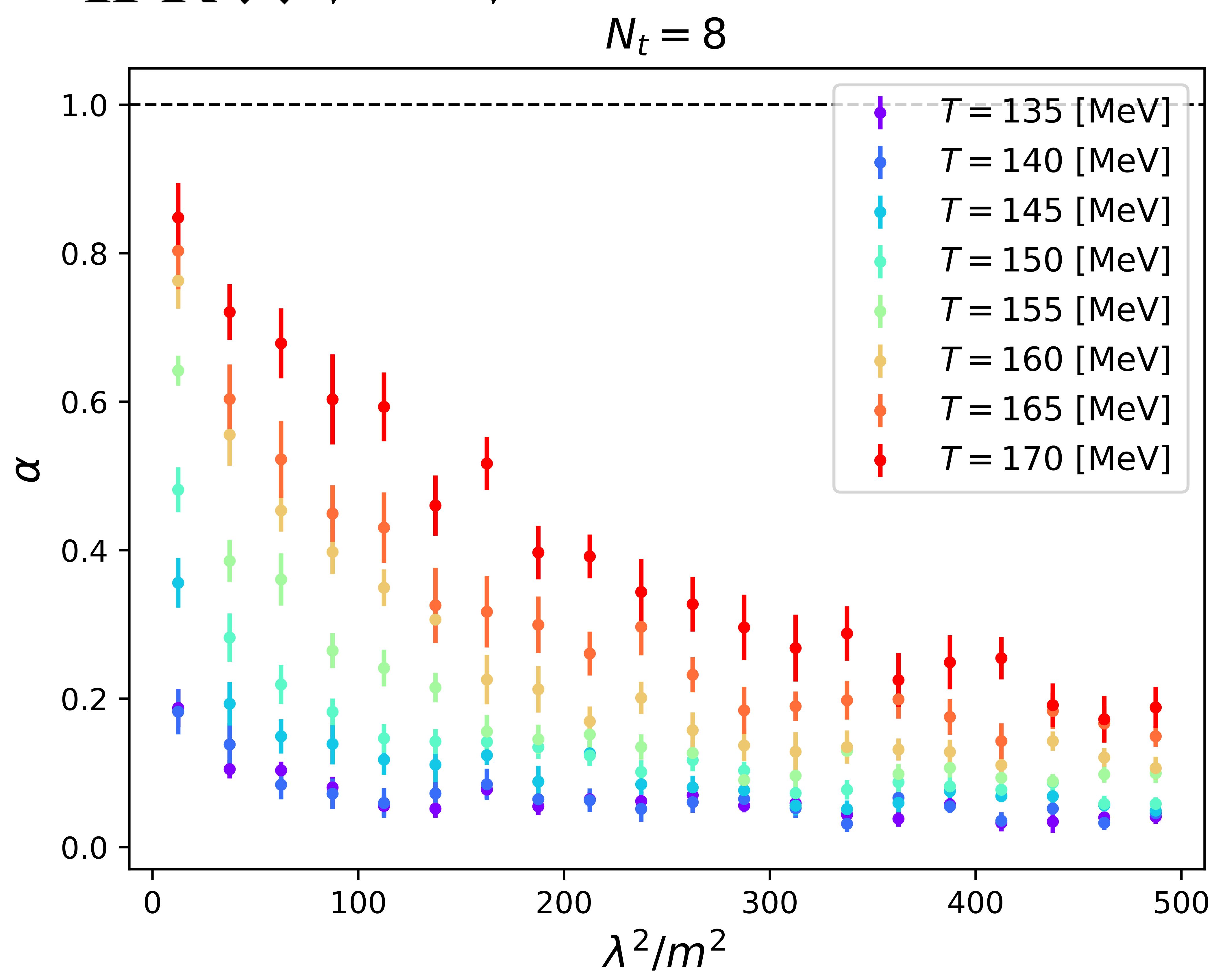
Localisation

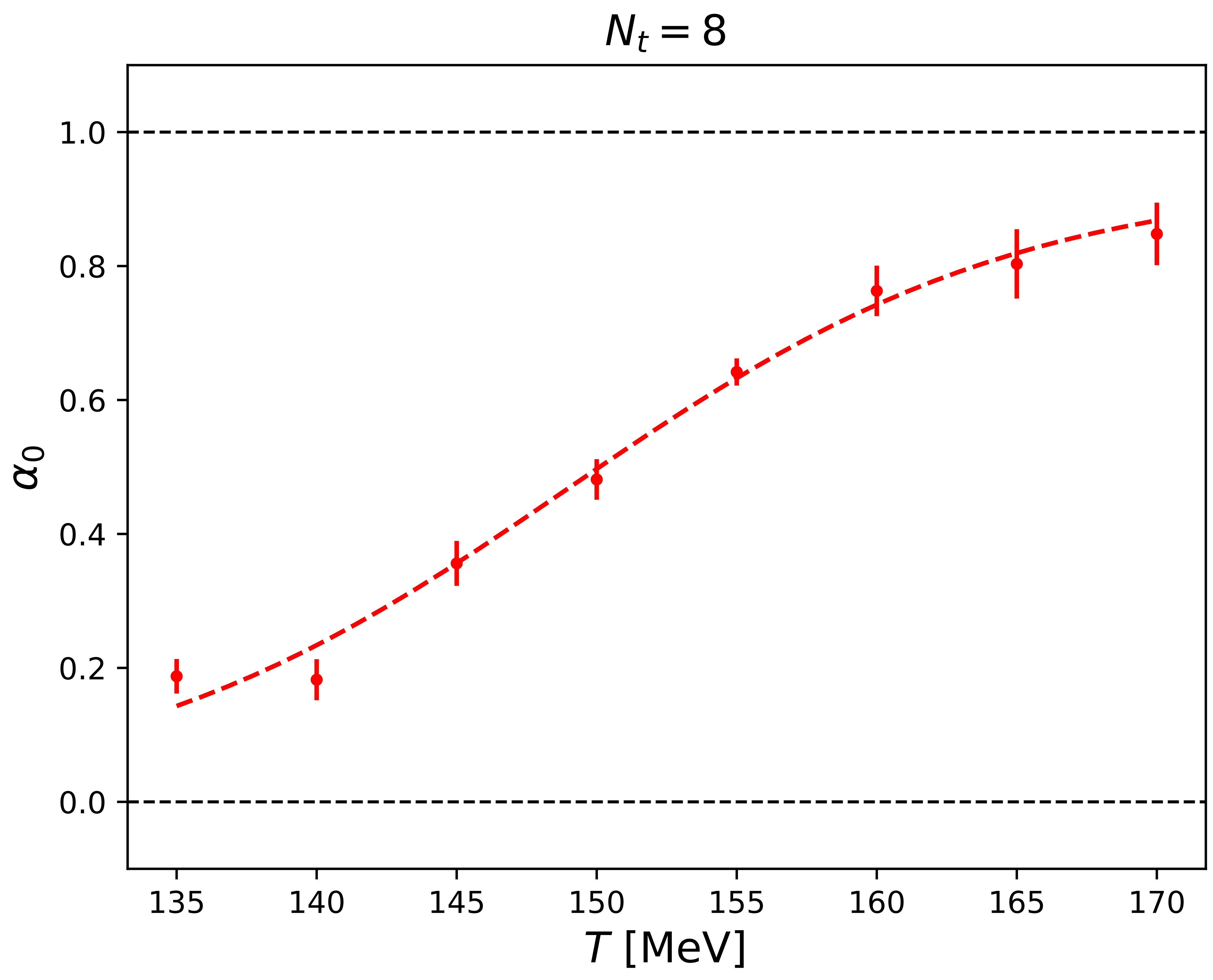
$N_t = 8$



Volume scaling

$\mathrm{IPR} \times V \sim V^{\alpha}$





$T_l(N_t = 8) \approx 150$ MeV: consistent with $T_c(N_t = 8)$

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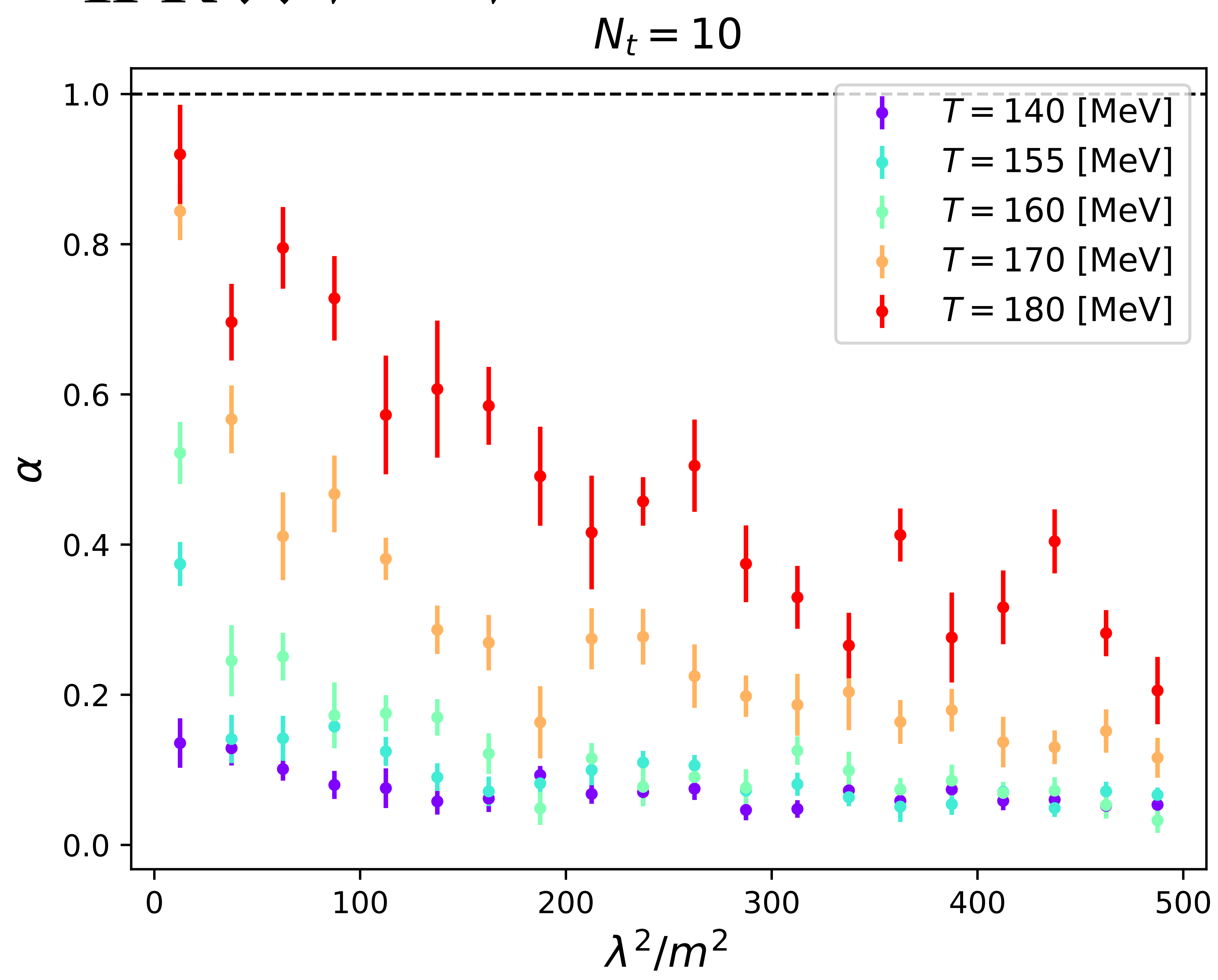
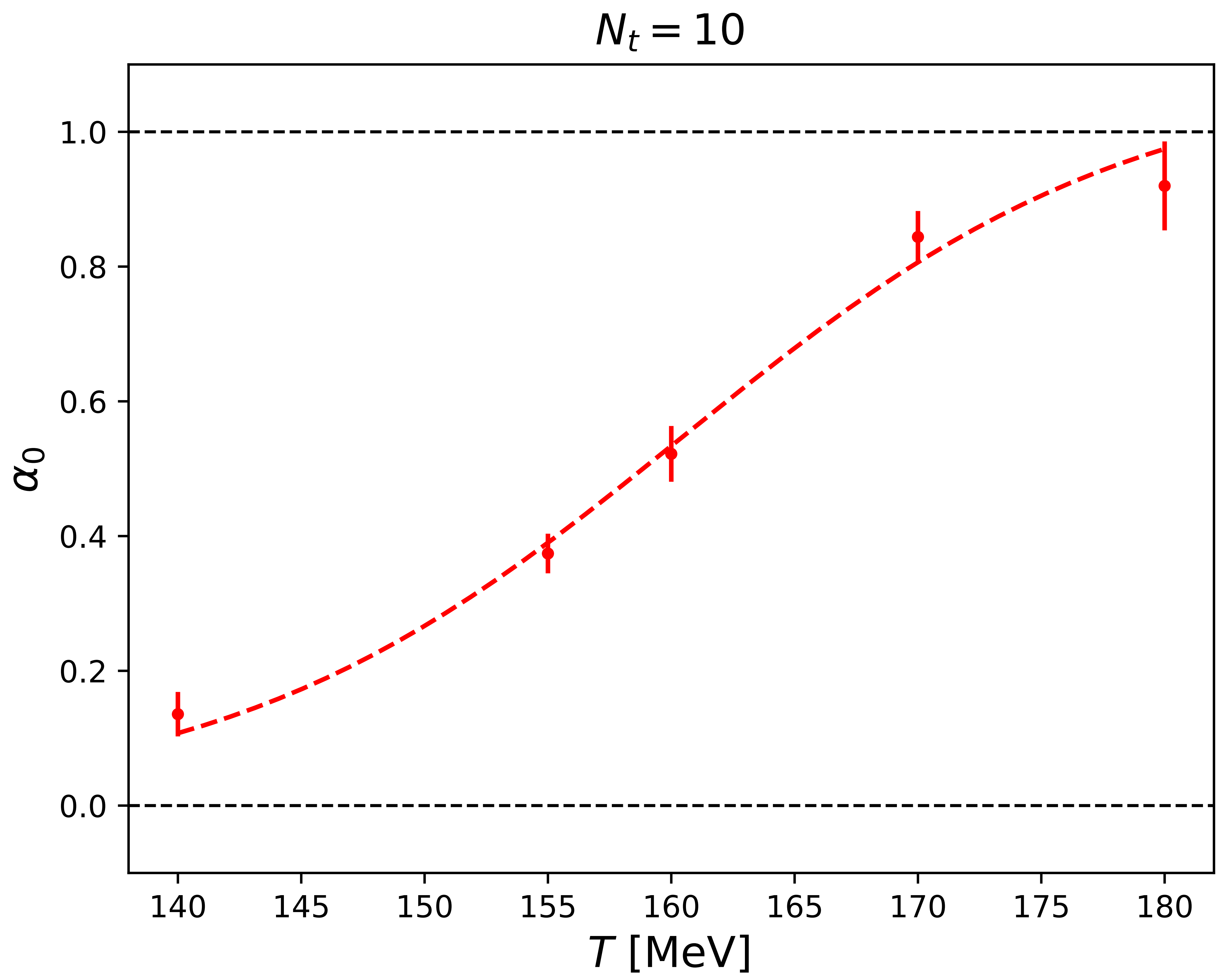




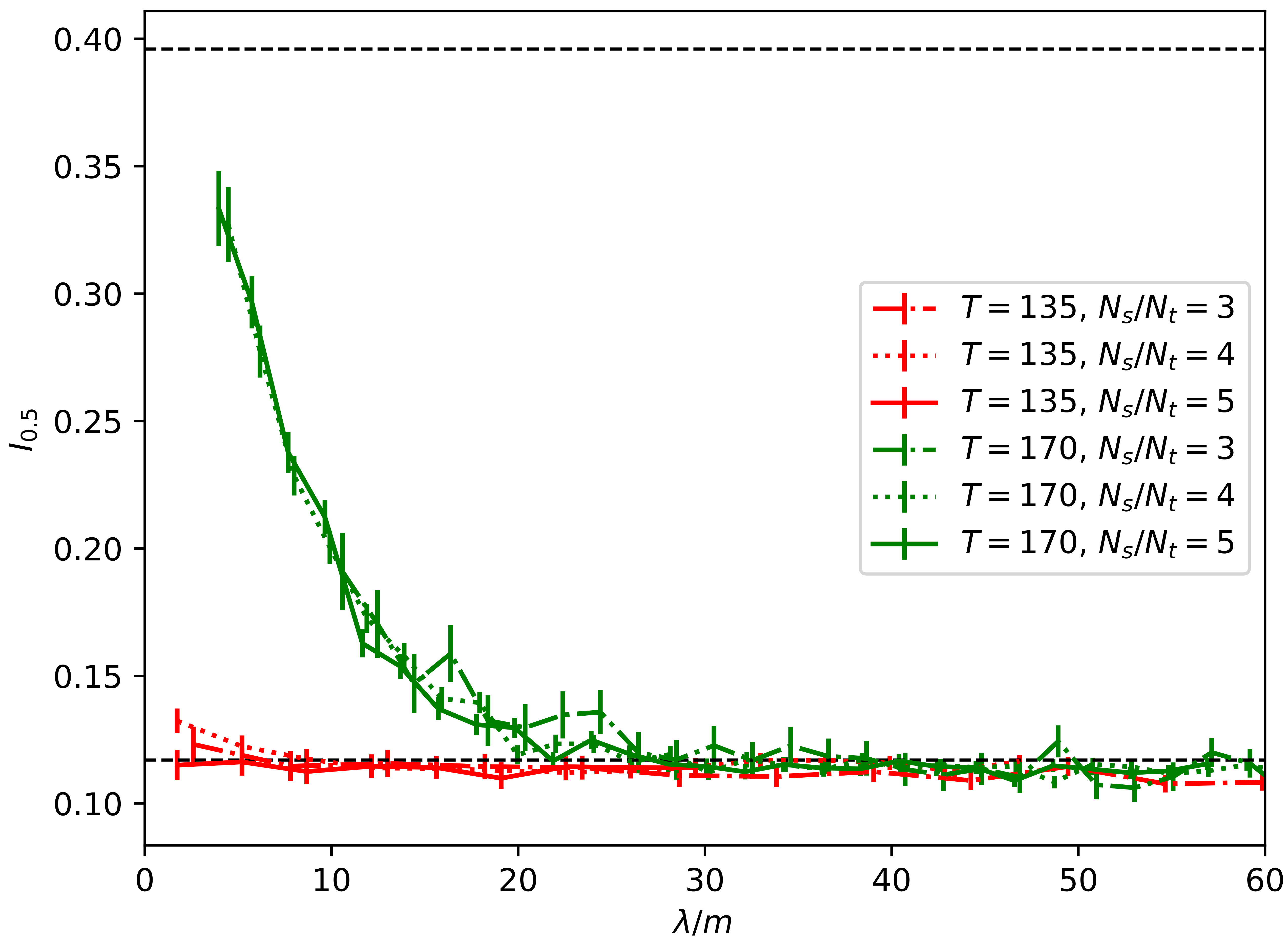
$T_l(N_t = 10) \approx 160$ MeV: consistent with $T_c(N_t = 10)$

Unfolded level spacing distribution

- Unfolding $x_i = \int^{\lambda_i} d\lambda \rho(\lambda) \quad s_i = x_{i+1} - x_i$
- Unfolded level spacing distribution (ULSD):

$$p(s, \lambda) = \frac{\langle \sum_n \delta(\lambda - \lambda_n)\delta(s - s_n) \rangle}{\rho(\lambda)}$$

- Integrated ULSD $I_{s_0} = \int_0^{s_0} ds\ p(s, \lambda, N_s)$
  - Localised: $I_{0.5}^{\text{Poisson}} \approx 0.396$
  - Delocalised: $I_{0.5}^{\text{RMT}} \approx 0.117$



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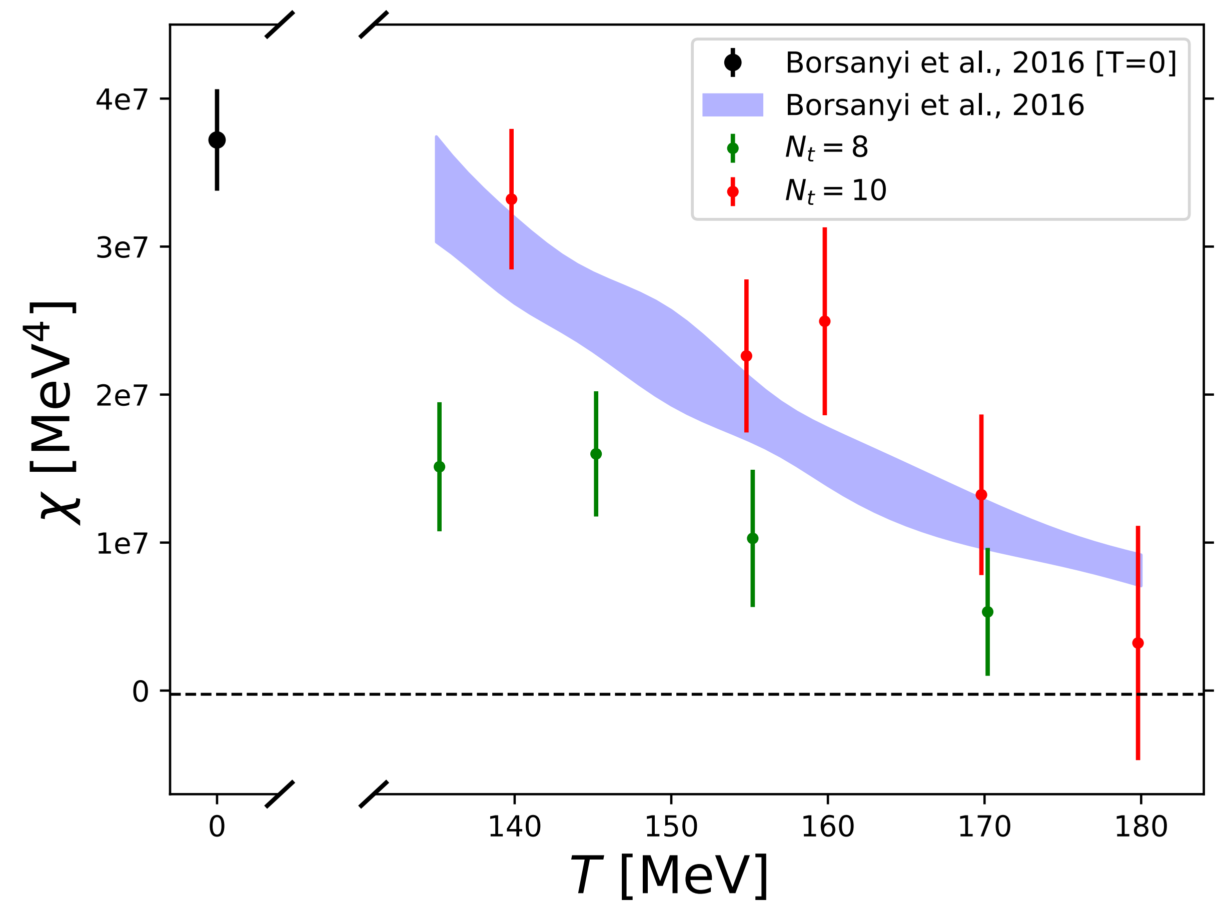

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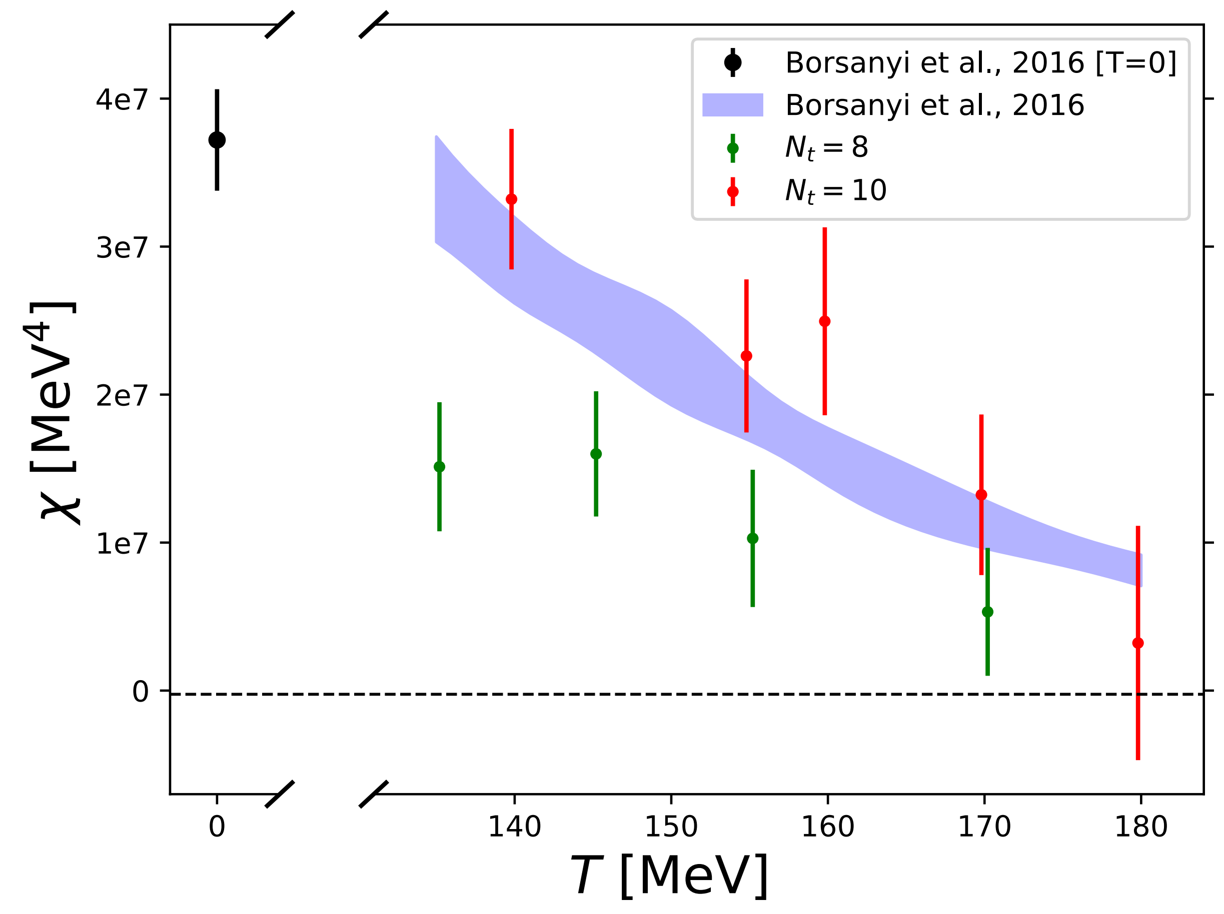

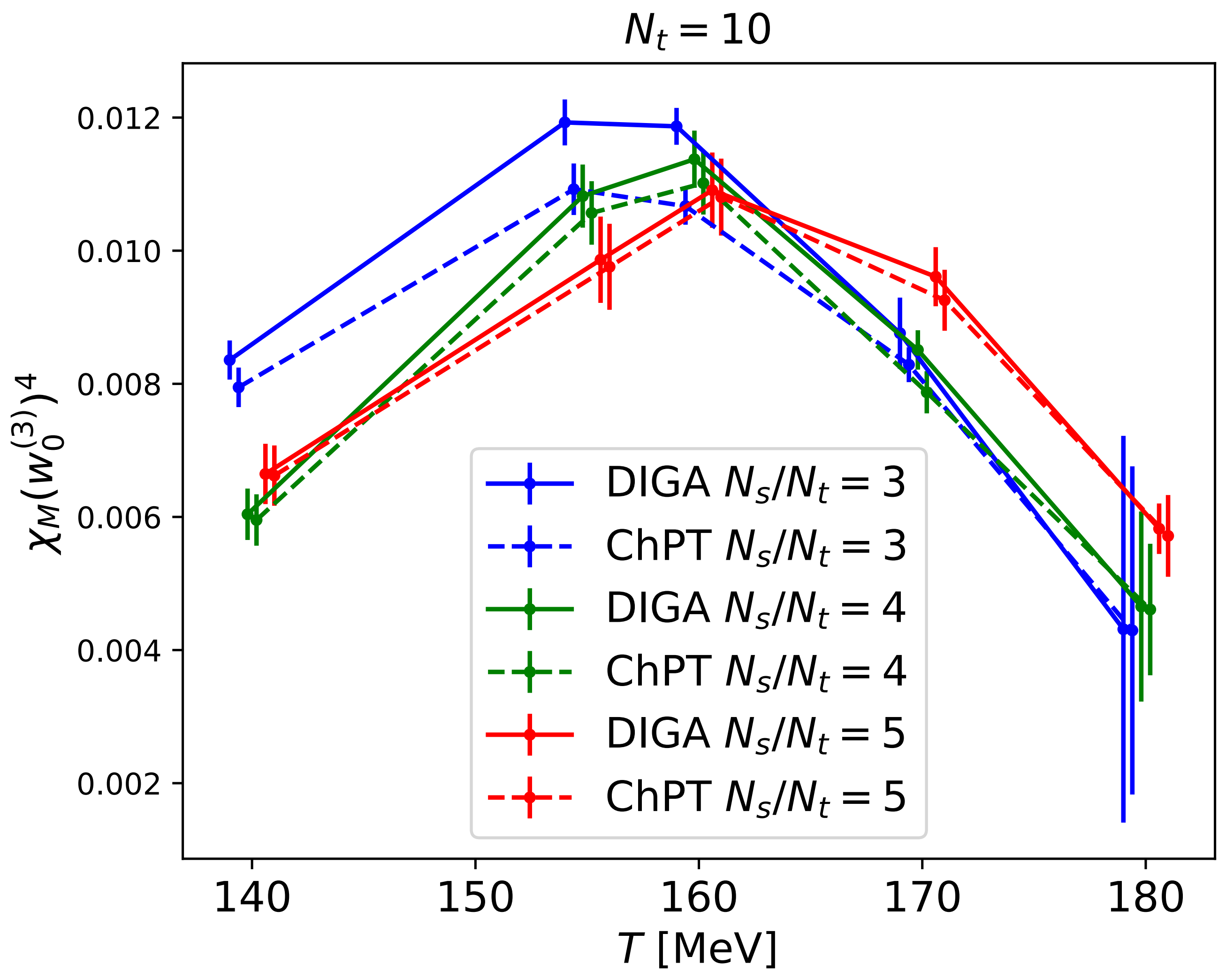

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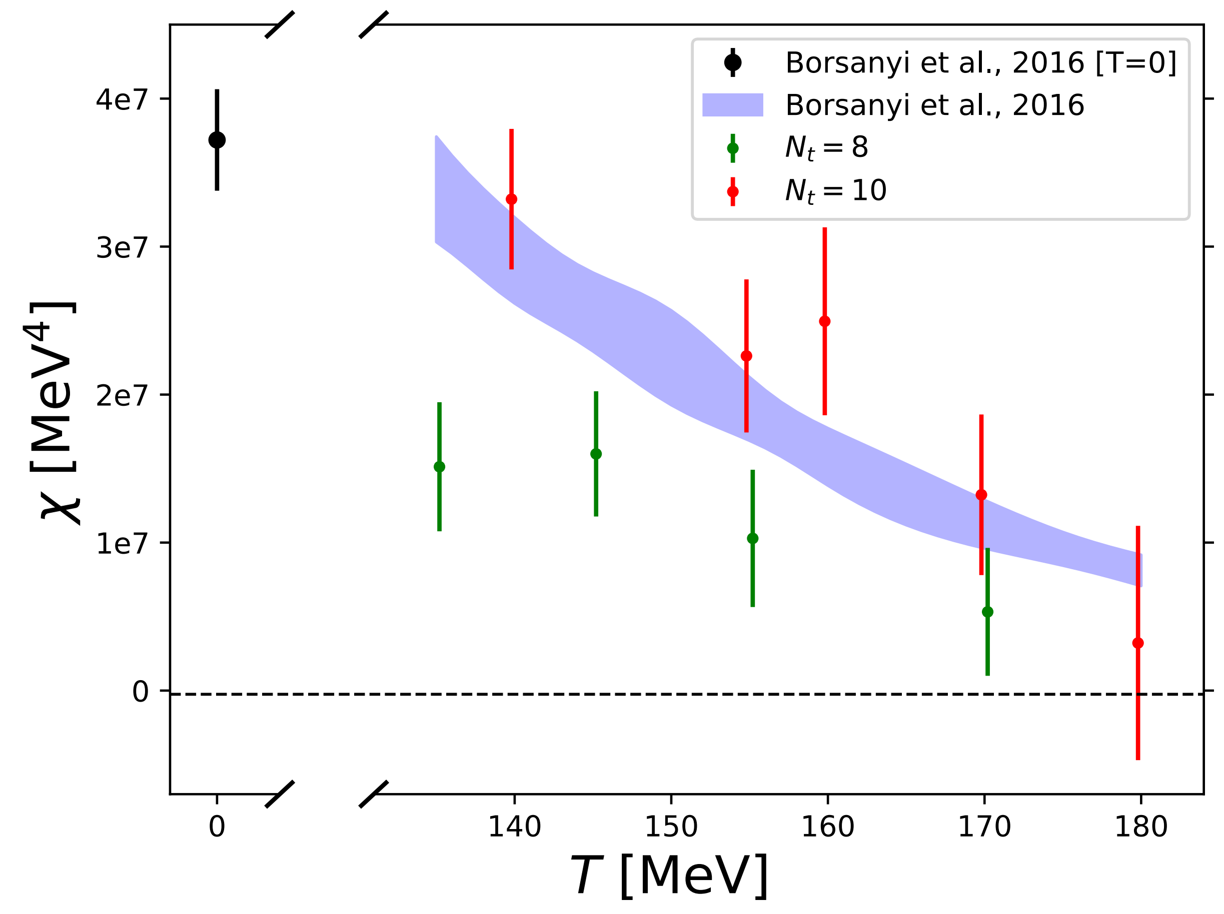

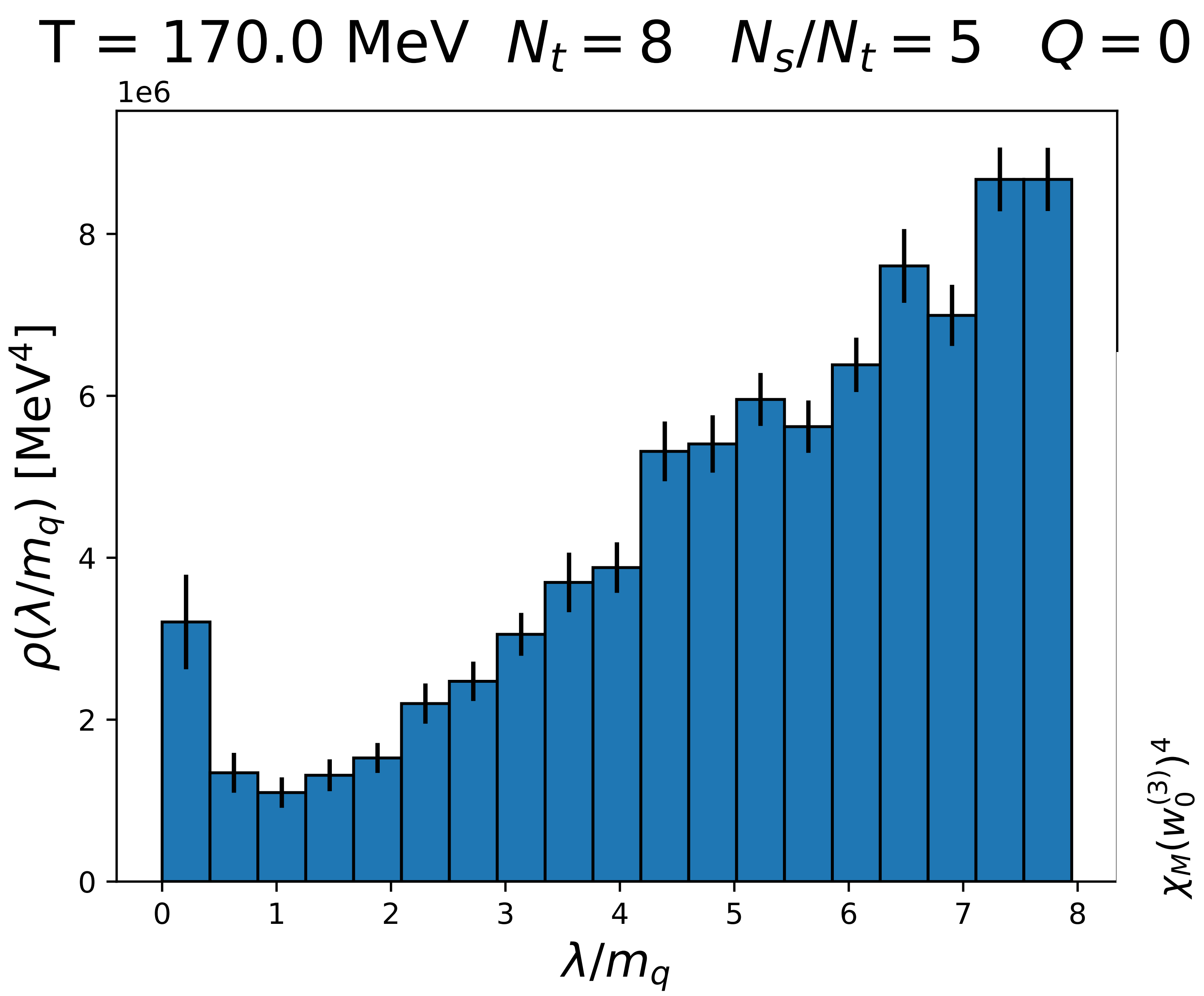

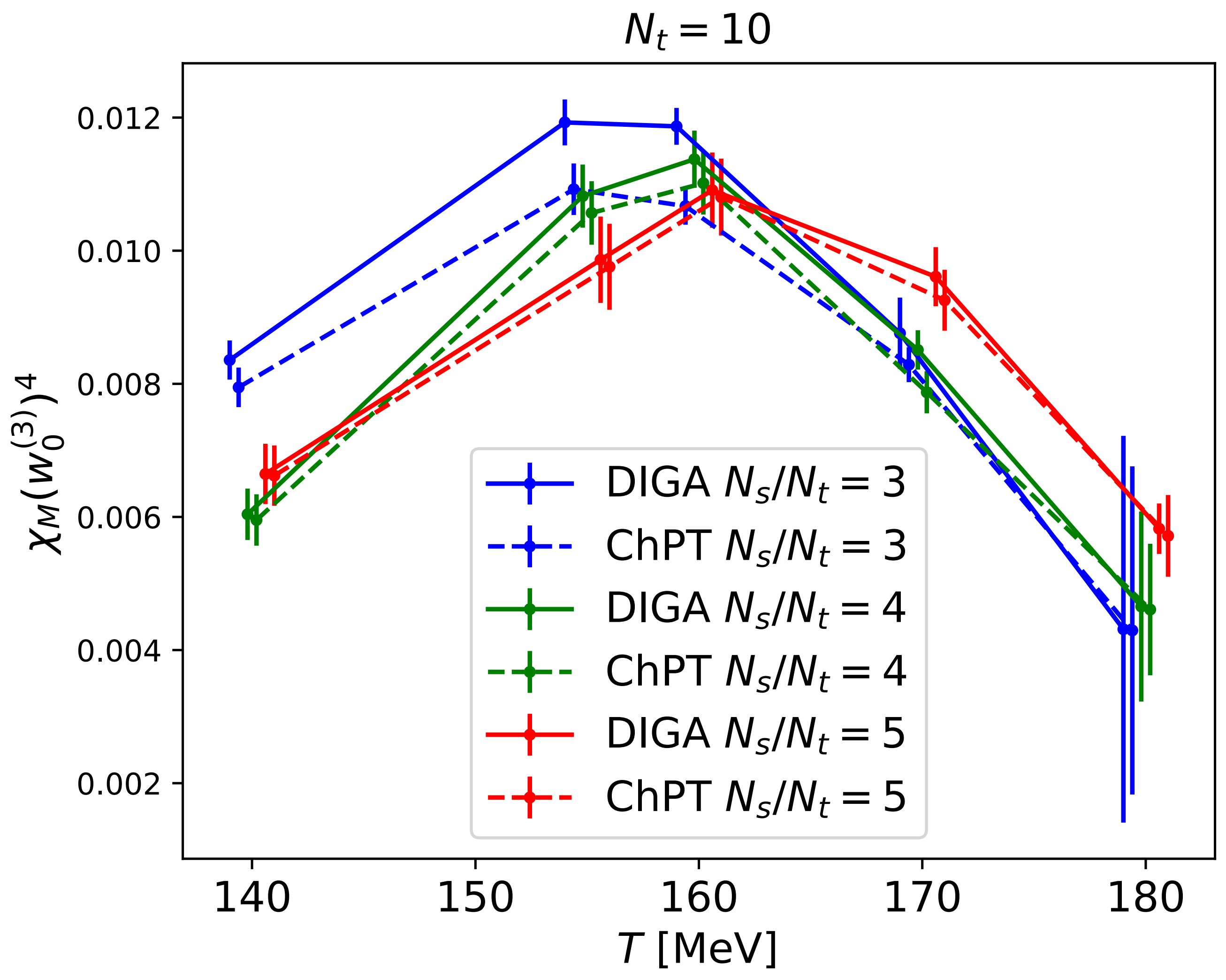

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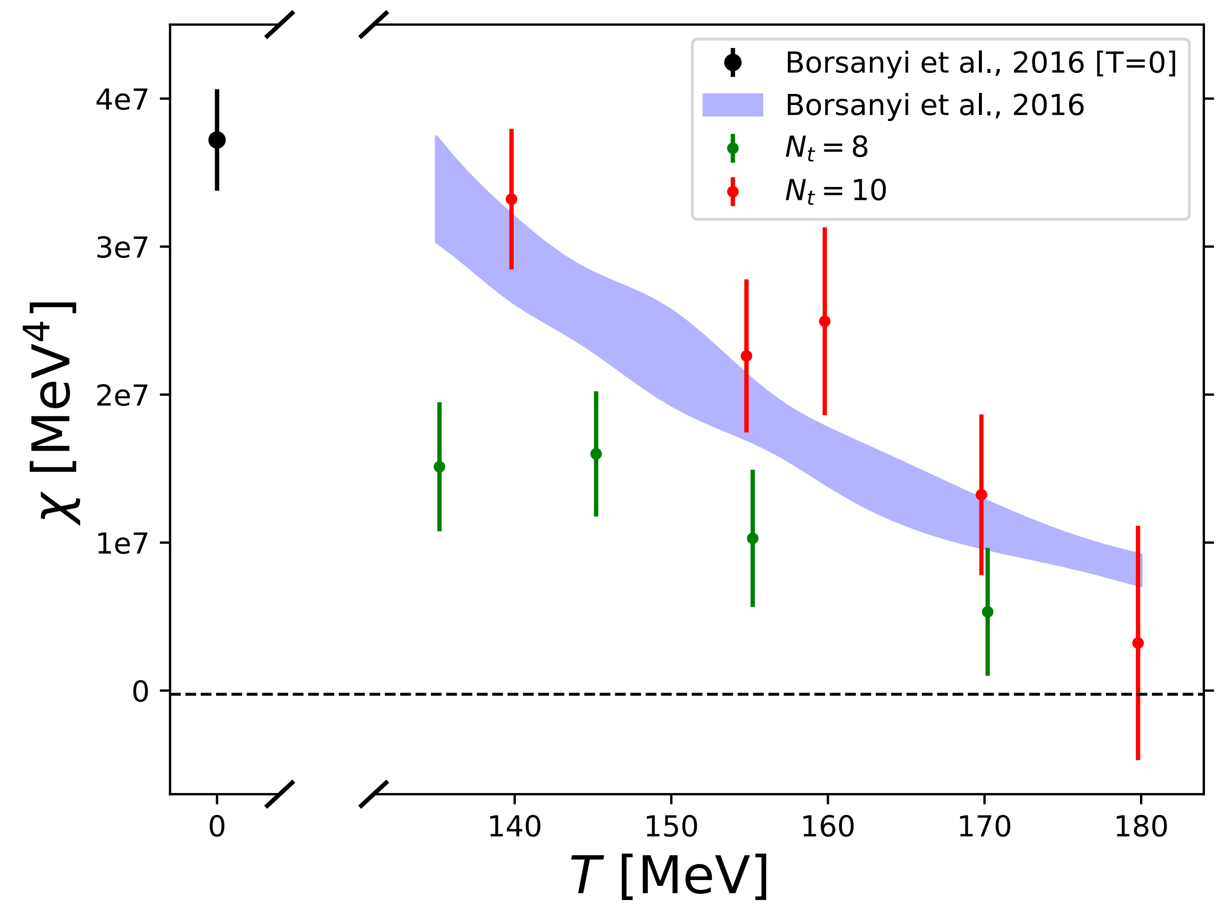

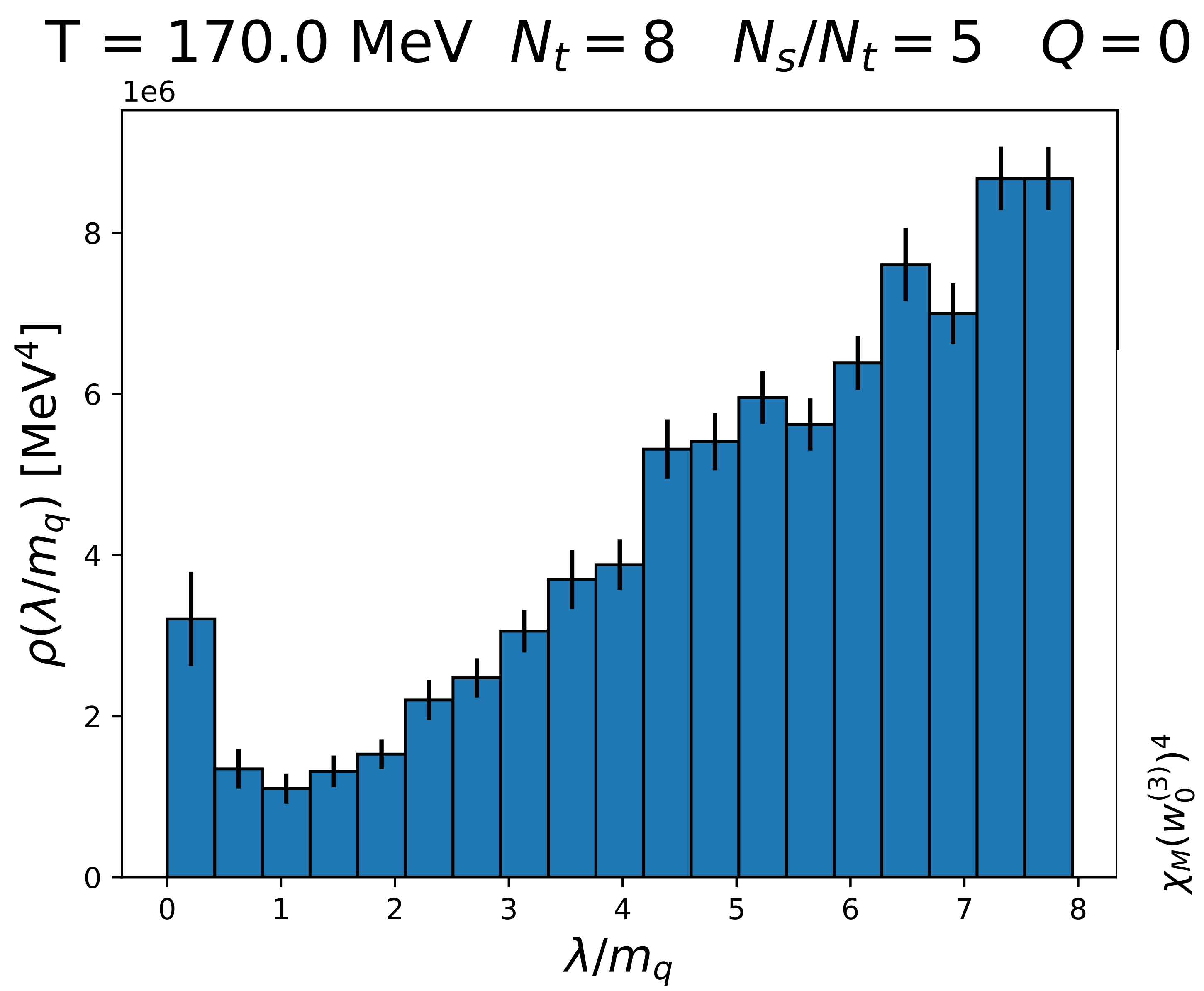

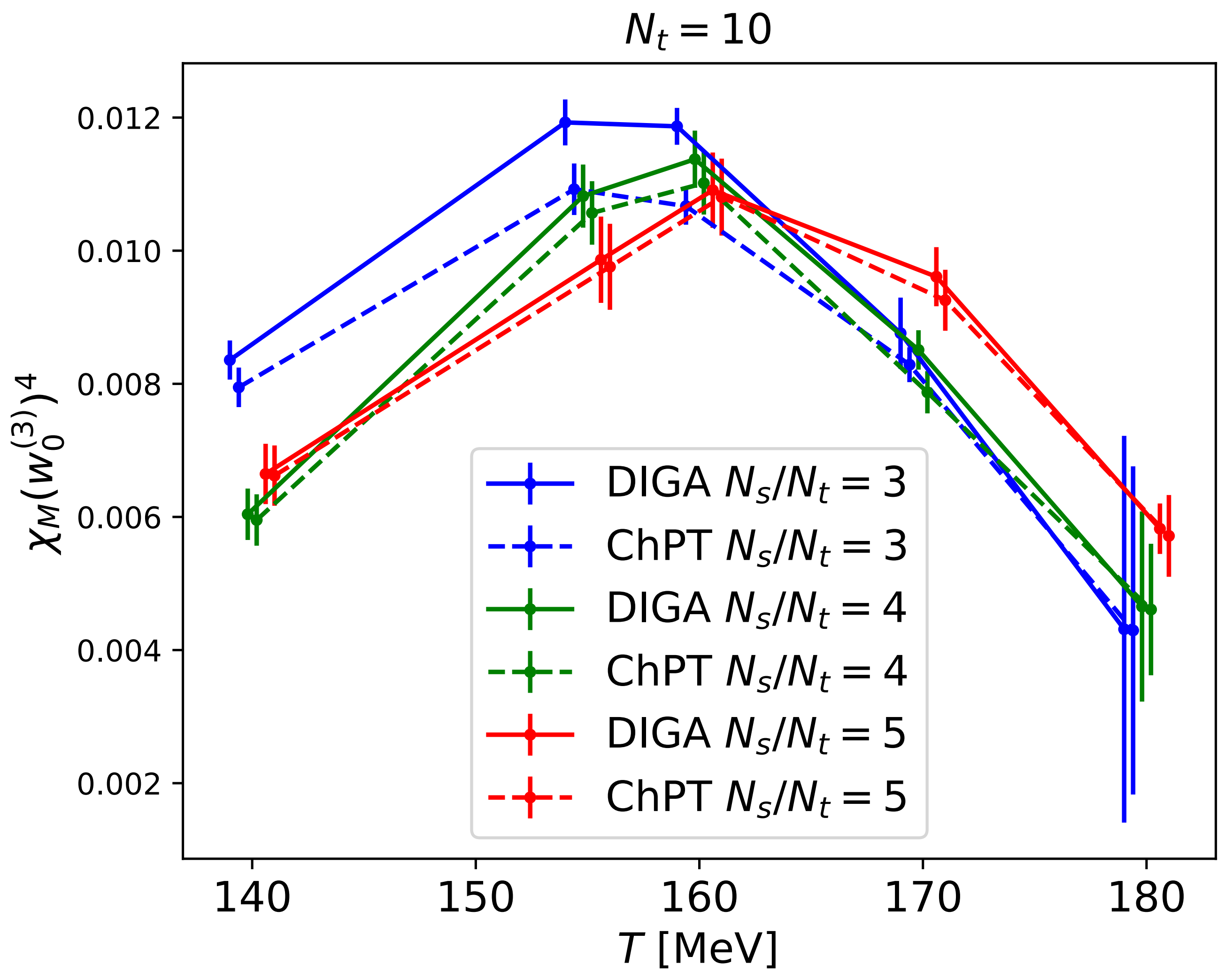

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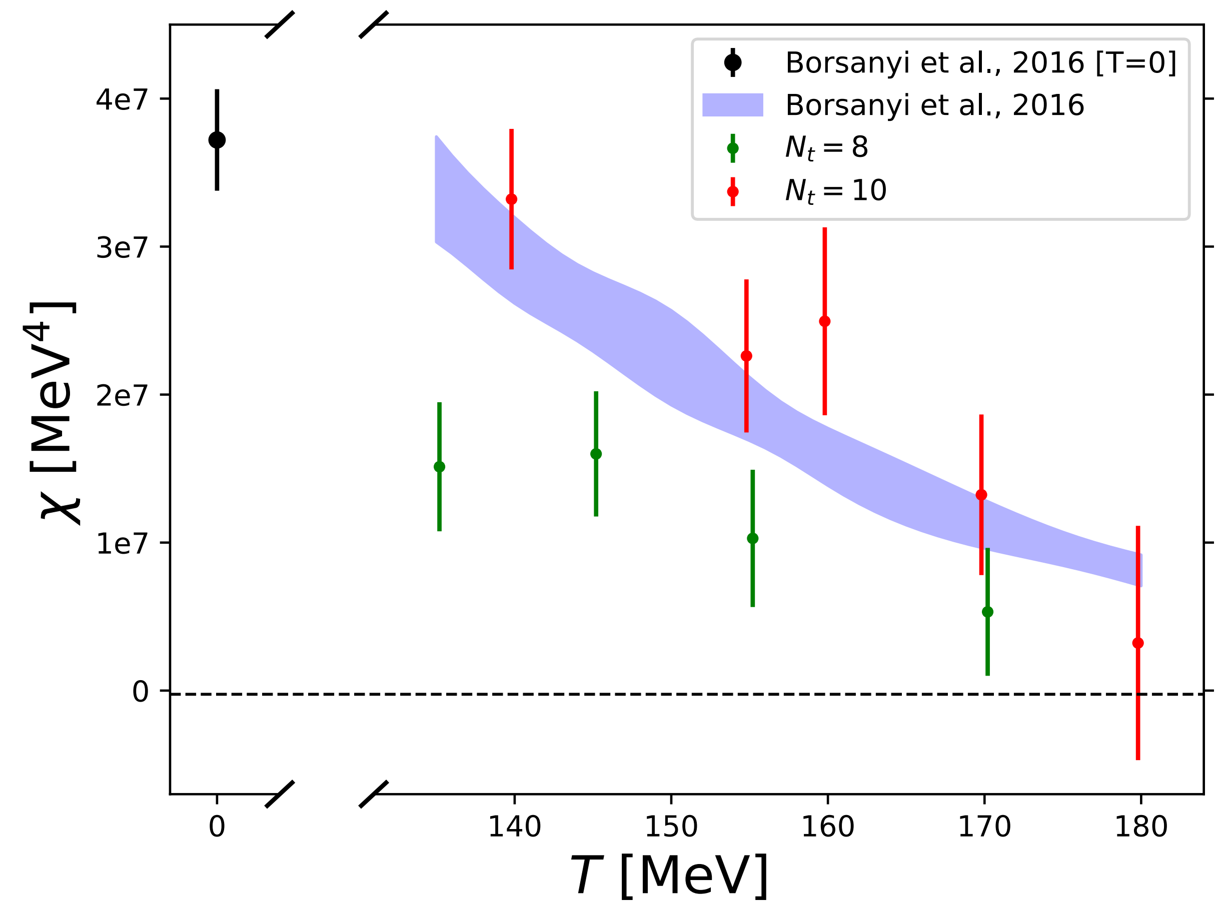
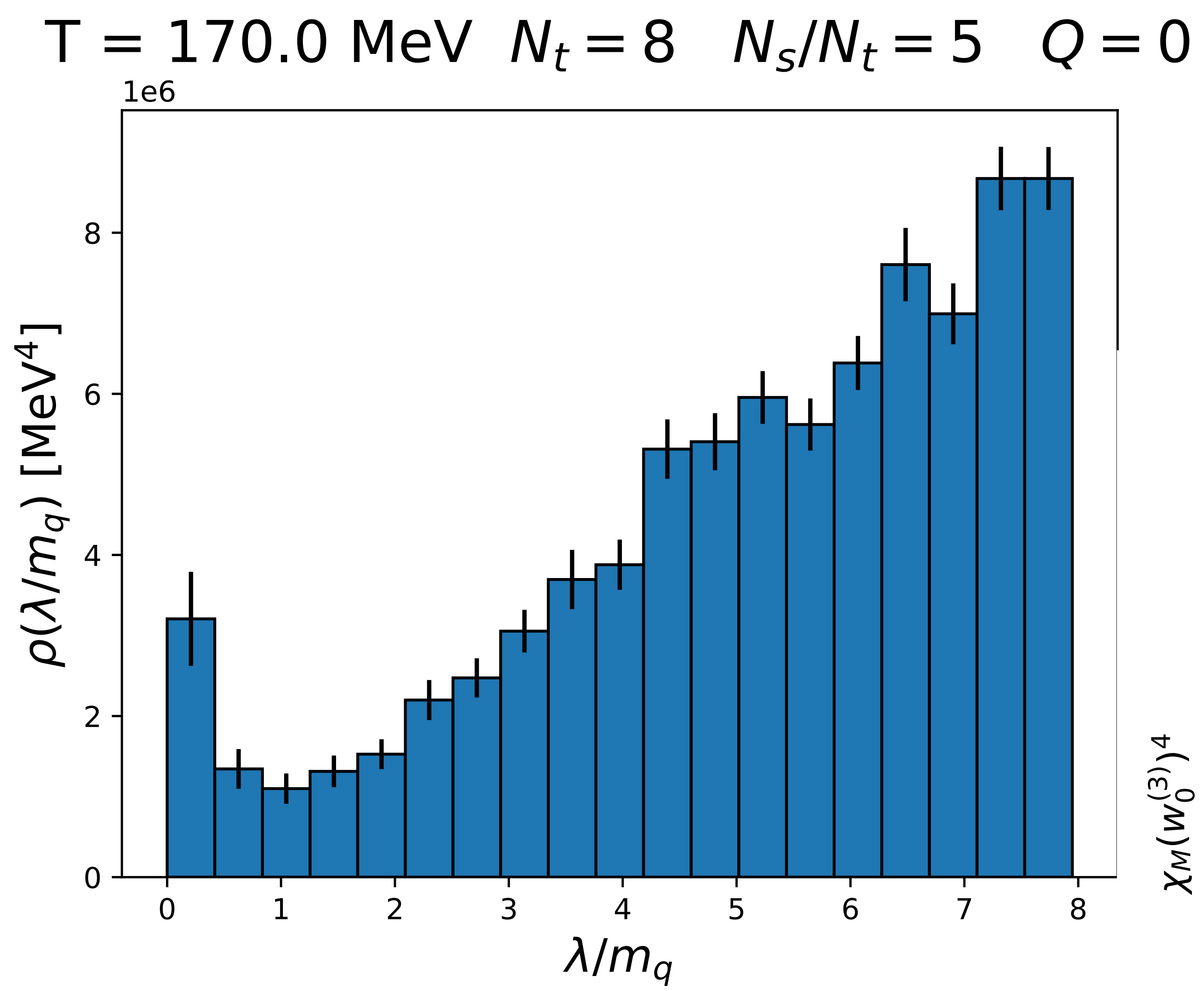
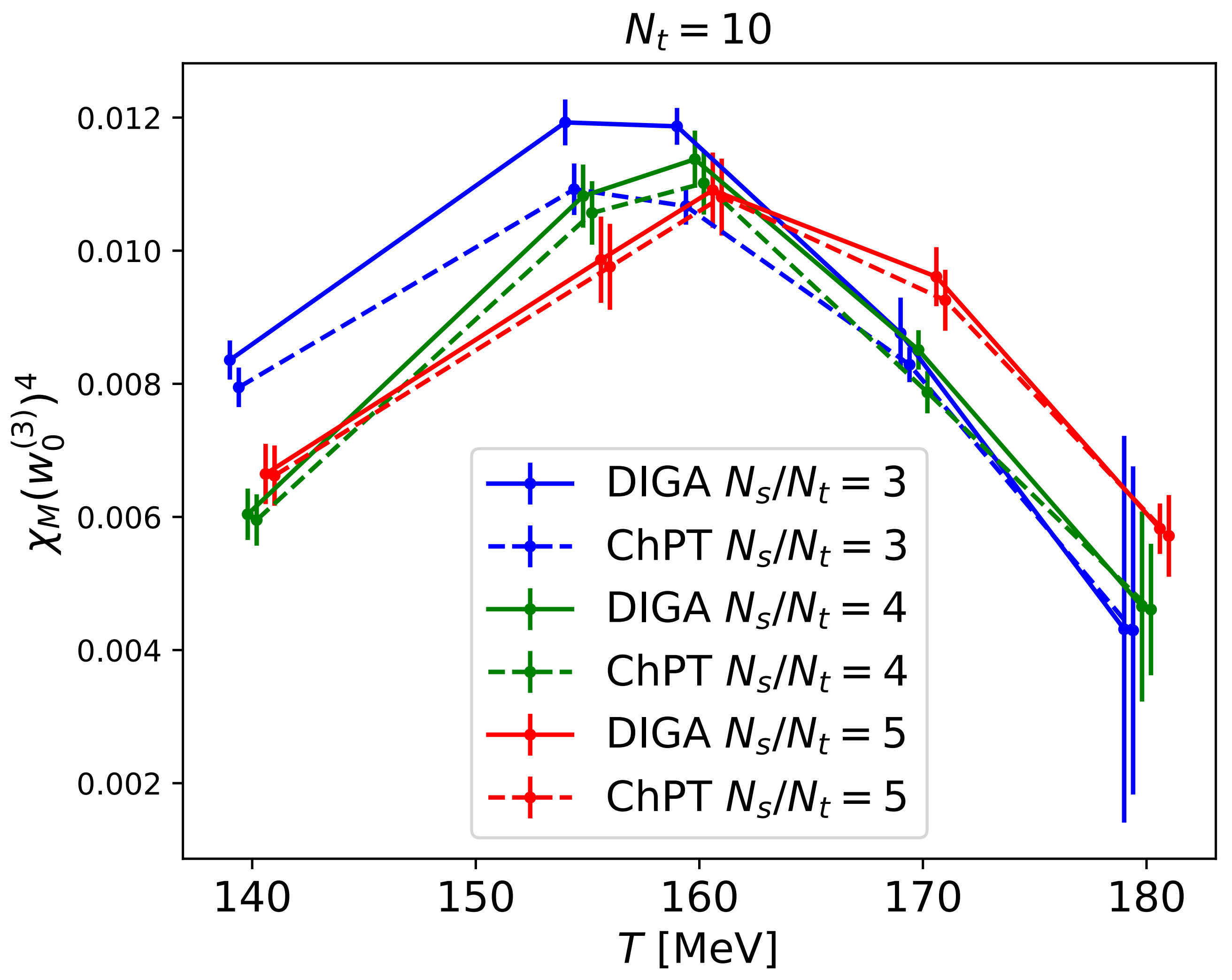




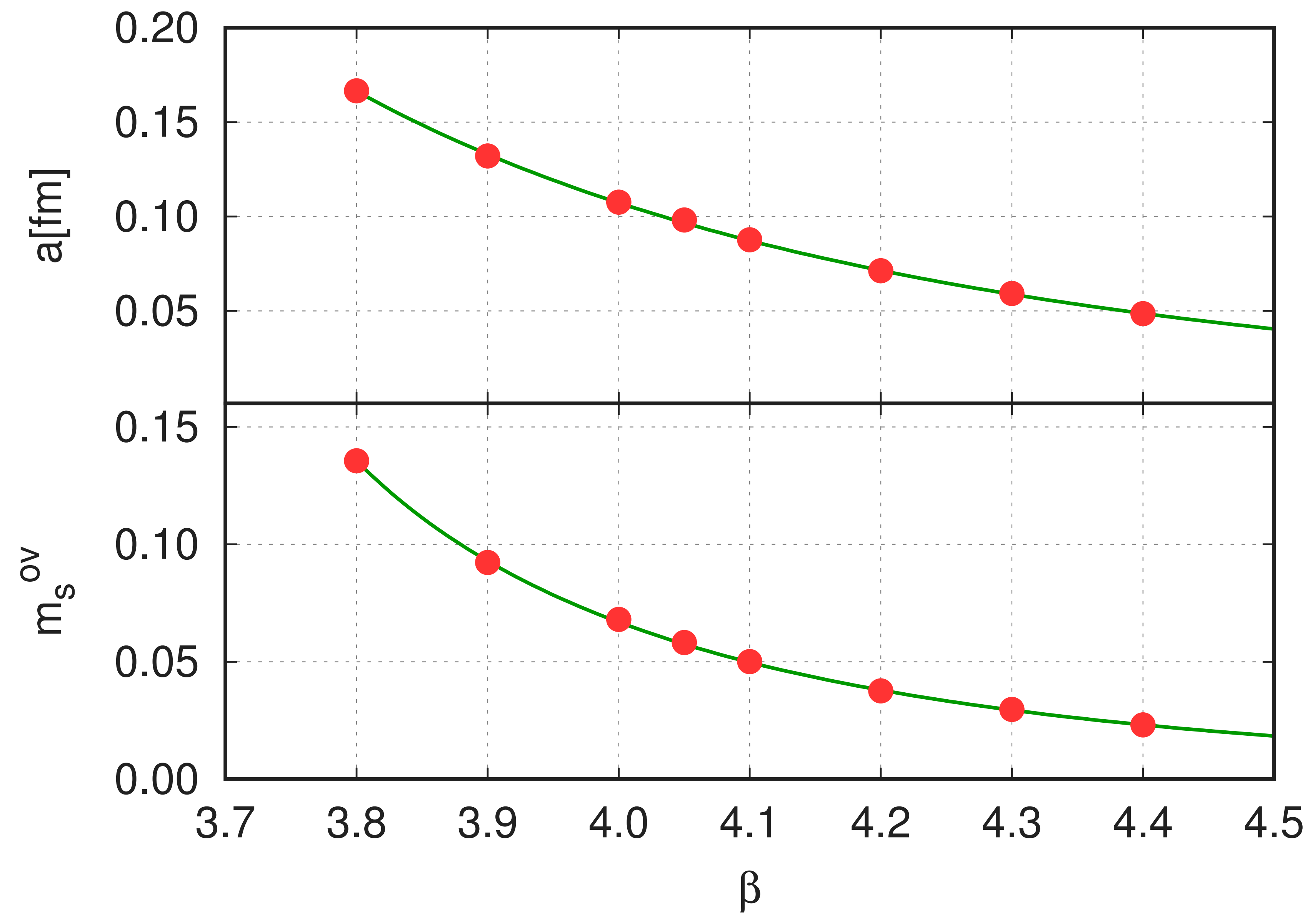


Thank you for your attention!

Backup

Lattice details, scale setting

Scale setting from simulations with large m_π



- Simulations are done along the LCP
- Scale setting: require $T = 0$ simulations
- $N_f = 3$ staggered simulations, $T = 0$, $w_0^{(3)} = 0.153(1)\ \mathrm{fm}, m_\pi^{(3)} = 712(5)\ \mathrm{MeV}$
- $N_f = 3$ overlap simulations, $T = 0$, at each β tune m_s^{ov} to have $m_\pi w_0 \equiv m_\pi^{(3)} w_0^{(3)}$
- $N_f = 2 + 1$ overlap simulations, $T \neq 0$: $m_s = m_s^{\mathrm{ov}}, \quad m_{ud} = R m_s^{\mathrm{ov}}, \quad a = w_0^{(3)}/w_0^{\mathrm{ov}}$
- Physical point: $m_{ud} = m_{ud}^{(phys)}, \quad m_s = m_s^{(phys)}$

[Borsanyi et al., 2016]

Implementing odd number of flavours

Exploiting $Q = \mathrm{const}$

- Monte Carlo: determinant of a hermitian operator $H^2 = D_{\mathrm{ov}} D_{\mathrm{ov}}^{\dagger}$: $N_f = 2$
- To simulate $\underline{N_f = 1}$ (strange quark): need to take the square root
- Chirality projectors: $P_{\pm} = \dfrac{1 \pm \gamma_5}{2}$, $H_{\pm}^2 = P_{\pm} H^2 P_{\pm}$
- Fixed topology $Q = \mathrm{const}$:
  $\det H^2 \sim \det H_+^2 \det H_-^2 \sim (\det H_+^2)^2 \sim (\det H_-^2)^2$
- Take $\det H_+^2$ or $\det H_-^2$