Constraints on the Dirac spectrum and the fate of $\mathrm{U}(1)_A$ in the chirally symmetric phase of QCD

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QCD in the chiral limit

QCD close to $N_f = 2$ chiral limit $m_{u,d} \to 0$, approximate chiral symmetry

$$\mathrm{U}(2)_L \times \mathrm{U}(2)_R = \underbrace{\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R}_{\to \mathrm{SU}(2)_V \text{ @low T}} \times \underbrace{\mathrm{U}(1)_A}_{\text{anomalous}} \times \mathrm{U}(1)_V$$

Fate of $\mathrm{U}(1)_A$ in the chiral limit in the symmetric phase still open question

- Believed to determine order of the transition [Pisarski, Wilczek (1984)], but
  - relation not so tight [Pelissetto, Vicari (2013), Fejős (2022), Bernhardt, Fischer (2023), Pisarski, Rennecke (2024)]
  - no sign of 1st order for $N_f = 3$ [Cuteri, Philipsen, Sciarra (2021)]
- $\mathrm{U}(1)_A$ part of the extended symmetry claimed to characterise QCD right above the crossover [Glozman, Philipsen, Pisarski (2022)]

Theoretical arguments both pro/con *effective* restoration

$$\langle G[e^{i\alpha\gamma_5}\Psi, \bar{\Psi}e^{i\alpha\gamma_5}]\rangle - \langle G[\Psi, \bar{\Psi}]\rangle \underset{m\to 0}{\to} 0$$

[Cohen (1996), Evans, Hsu, Schwetz (1996), Lee, Hatsuda (1996), Cohen (1997), Aoki, Fukaya, Taniguchi (2012), Kanazawa, Yamamoto (2016), Azcoiti (2023)]

Numerical lattice results also contradictory [HotQCD (2019), JLQCD (2021)]

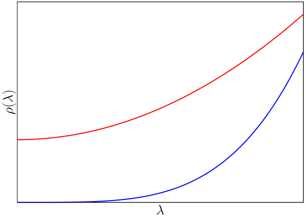
Chiral symmetry and Dirac spectrum

Constraints on the Dirac spectrum from $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ restoration can give us first-principles information about the fate of $\mathrm{U}(1)_A$ [Cohen (1997)]

Used in [Aoki, Fukaya, Taniguchi (2012)] to argue effective $\mathrm{U}(1)_A$ restoration, assuming spectral density vanishes at $\lambda = 0$

$$-\langle\bar{\psi}\psi\rangle = \int_0^{\text{“}\infty\text{”}} d\lambda \, \frac{2m}{\lambda^2 + m^2} \, \rho(\lambda; m)$$

$$\rho(\lambda; m) = \lim_{\mathrm{V}\to\infty} \tfrac{\mathrm{T}}{\mathrm{V}} \left\langle \textstyle\sum_{n,\lambda_n \neq 0} \delta(\lambda - \lambda_n) \right\rangle$$



Banks–Casher relation: $\lim_{m\to 0} |\langle\bar{\psi}\psi\rangle| = \pi\rho(0^+; 0)$ [Banks, Casher (1980)]

low T: $\langle\bar{\psi}\psi\rangle_{m\to 0} \neq 0$, expect $\rho(\lambda \simeq 0; m) \neq 0$ — find it

high T: $\langle\bar{\psi}\psi\rangle_{m\to 0} = 0$, expect $\rho(\lambda \simeq 0; m) \simeq 0$ — get singular peak!

[Edwards *et al.* (1999), Cossu *et al.* (2013), Alexandru, Horváth (2015), Dick *et al.* (2015), Brandt *et al.* (2016), Tomiya *et al.* (2017)]
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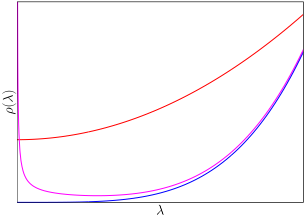

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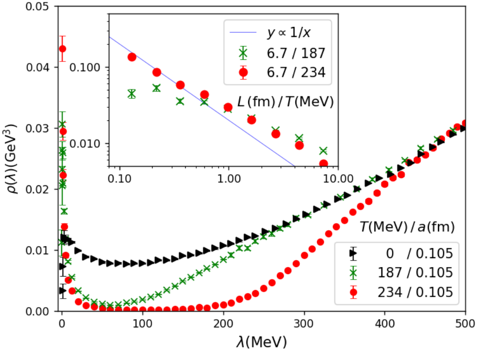
Dirac spectrum above T_c



Overlap spectrum on $N_f = 2 + 1$ lattice QCD configurations [Meng *et al.* (2023)]

tadpole-improved clover fermions and Symanzik gauge action, 1 stout smearing $\rho = 0.125$,

$a = 0.105\text{fm}$, $L = 5$ fm, $m_\pi \simeq 135$ MeV

- How can a singular peak fit in with chiral symmetry restoration?
- What does it do to $\mathrm{U}(1)_A$?

Chiral symmetry restoration and the fate of $\mathrm{U}(1)_A$

How does one characterise $\mathrm{SU}(2)_A$ restoration?

assumptions	conclusions
• $\langle F[A]\rangle$ analytic in m^2; $\langle \delta_A O\rangle \underset{m\to 0}{\rightarrow} 0$; ρ power series in $\lvert\lambda\rvert$ near $\lambda = 0$, or $\sim \lvert\lambda\rvert^\alpha$, $\alpha > 0$	$\mathrm{SU}(2)_A$ restoration $\Rightarrow$ $\mathrm{U}(1)_A$ restoration [Cohen (1997), Aoki, Fukaya, Taniguchi (2012), Kanazawa, Yamamoto (2016)]
• thermodynamic and chiral limit commute	$\mathrm{SU}(2)_A$ restoration $\not\Rightarrow$ $\mathrm{U}(1)_A$ restoration, $\mathrm{U}(1)_A$ broken by topological effects [Evans, Hsu, Schwetz (1996), Lee, Hatsuda (1996)]
• free energy analytic in m^2; thermodynamic and chiral limit commute	$\mathrm{SU}(2)_A$ restoration $\Rightarrow$ $\mathrm{U}(1)_A$ restoration, unless $\rho \sim m^2\delta(\lambda)$ [Azcoiti (2023)]

What assumptions follow from first principles?

Symmetry restoration and susceptibilities

Local quantum field theory: symmetry restored $\Leftrightarrow$ symmetry-related local correlators become equal in the symmetric limit

$$\lim_{m\to 0}\left(\left\langle F'_{i_1}(x_1)\ldots F'_{i_n}(x_n)\right\rangle - \left\langle F_{i_1}(x_1)\ldots F_{i_n}(x_n)\right\rangle\right) = 0$$

No massless excitations, finite correlation length within symmetric phase $\Rightarrow$ finite (non-divergent) susceptibilities, equal if symmetry-related

$$\chi_{i_1\ldots i_n} = \lim_{\mathrm{V}\to\infty}\frac{1}{\mathrm{V}}\int d^4x_1\ldots\int d^4x_n\,\left\langle F_{i_1}(x_1)\ldots F_{i_n}(x_1)\right\rangle_c$$

$$\left|\lim_{m\to 0}\chi_{i_1\ldots i_n}\right| < \infty \qquad \lim_{m\to 0}\left(\chi'_{i_1\ldots i_n} - \chi_{i_1\ldots i_n}\right) = 0$$

Probing with external, partially quenched fermions should not alter symmetry restoration expect symmetric local correlators of dynamical and external fields symmetric

$\Rightarrow$ **expect** symmetric and finite susceptibilities involving external fields

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Scalar and pseudoscalar bilinears

	isosinglet	isotriplet
scalar	$S = \bar{\psi}\psi$	$\vec{S} = \bar{\psi}\vec{\sigma}\psi$
pseudoscalar	$P = \bar{\psi}\gamma_5\psi$	$\vec{P} = \bar{\psi}\vec{\sigma}\gamma_5\psi$

Irreducible $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ multiplets

$$O_{\mathcal{S}} = \begin{pmatrix} S \\ \mathrm{i}\vec{P} \end{pmatrix} \qquad O_{\mathcal{P}} = \begin{pmatrix} \mathrm{i}P \\ -\vec{S} \end{pmatrix}$$

Under chiral transformations

$$O_{\mathcal{S},\mathcal{P}} \xrightarrow{\mathcal{U}} \mathcal{R}(\mathcal{U})O_{\mathcal{S},\mathcal{P}} \qquad \mathcal{R} \in \mathrm{SO}(4)$$

Corresponding susceptibilities expressible in terms of the spectrum of D only, constraints result from finiteness implied by symmetry restoration

Regularise theory on the lattice, chiral symmetry problematic but GW fermions [Ginsparg, Wilson (1982)] have exact $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ [Lüscher (1998)]

Generating function

Include source terms for bilinears in the partition function

$$\begin{aligned}\mathcal{Z}(\mathrm{J}_{\mathcal{S}},\mathrm{J}_{\mathcal{P}};m) &= \int DA \int D\psi D\bar{\psi}\, e^{-S_{\rm eff}[A]-\bar{\psi}(\not{D}[A]+m)\psi - K[\psi,\bar{\psi},A;\mathrm{J}_{\mathcal{S}},\mathrm{J}_{\mathcal{P}}]}\\ S_{\rm eff} &= \text{gauge and massive fermion contributions}\\ K[\psi,\bar{\psi},A;\mathrm{J}_{\mathcal{S}},\mathrm{J}_{\mathcal{P}}] &= j_S S + \vec{\jmath}_P\cdot(\mathrm{i}\vec{P}) + j_P\,(\mathrm{i}P) + \vec{\jmath}_S\cdot(-\vec{S})\\ &= \mathrm{J}_{\mathcal{S}}\cdot O_{\mathcal{S}} + \mathrm{J}_{\mathcal{P}}\cdot O_{\mathcal{P}}\end{aligned}$$

Generating function of scalar and pseudoscalar susceptibilities

$$\mathcal{W}(\mathrm{J}_{\mathcal{S}},\mathrm{J}_{\mathcal{P}};m) = \lim_{\mathrm{V}\to\infty}\frac{\mathrm{T}}{\mathrm{V}}\ln\mathcal{Z}(\mathrm{J}_{\mathcal{S}},\mathrm{J}_{\mathcal{P}};m)$$

Under chiral transformations $\mathcal{W}(\mathrm{J}_{\mathcal{S}},\mathrm{J}_{\mathcal{P}};m)\xrightarrow{\mathcal{U}^{-1}}\mathcal{W}(\mathcal{R}\mathrm{J}_{\mathcal{S}},\mathcal{R}\mathrm{J}_{\mathcal{P}};m)$

Using $\mathcal{Z}$ function of $j_S + m$ only **+** chiral symmetry of $m \equiv 0$ theory

$$\mathcal{W}(\mathrm{J}_{\mathcal{S}},\mathrm{J}_{\mathcal{P}};m) = \hat{\mathcal{W}}\big((m\hat{\jmath}_S+\mathrm{J}_{\mathcal{S}})^2, \mathrm{J}_{\mathcal{P}}^2, 2(m\hat{\jmath}_S+\mathrm{J}_{\mathcal{S}})\cdot\mathrm{J}_{\mathcal{P}}\big)$$

Symmetry restoration in scalar/pseudoscalar sector

$$\mathcal{W}(\mathrm{J}_{\mathcal{S}}, \mathrm{J}_{\mathcal{P}}; m) = \sum_{n_u, n_w, n_{\tilde{u}} \geq 0} \frac{u^{n_u} w^{n_w} \tilde{u}^{n_{\tilde{u}}}}{n_u! n_w! n_{\tilde{u}}!} \mathcal{A}_{n_u n_w n_{\tilde{u}}}(m^2)$$

$$u = 2mj_S + \mathrm{J}_{\mathcal{S}}^2 , \qquad w = \mathrm{J}_{\mathcal{P}}^2 , \qquad \tilde{u} = 2(mj_P + \mathrm{J}_{\mathcal{S}} \cdot \mathrm{J}_{\mathcal{P}})$$

$\mathcal{A}_{n_u n_w n_{\tilde{u}}}$ = linear combination of isotriplet susceptibilities,
finite susceptibilities as $m \to 0 \Longrightarrow$ finite $\mathcal{A}_{n_u n_w n_{\tilde{u}}}(0)$, $\forall n_{u,w,\tilde{u}}$

$$\partial_{m^2} \mathcal{W} = \tfrac{1}{2m} \partial_m \mathcal{W} = \tfrac{1}{2m} \partial_{j_S} \mathcal{W} = \left(1 + \tfrac{j_S}{m}\right) \partial_u \mathcal{W}$$

$$\partial_{m^2} \mathcal{W}|_{j_{S,P}=0} = \textstyle\sum_n \frac{u^{n_u} w^{n_w} \tilde{u}^{n_{\tilde{u}}}}{n_u! n_w! n_{\tilde{u}}!} \partial_{m^2} \mathcal{A}_{n_u n_w n_{\tilde{u}}}(m^2)|_{j_{S,P}=0} = \partial_u \mathcal{W}|_{j_{S,P}=0}$$

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$\Longrightarrow \mathcal{A}_{n_u n_w n_{\tilde{u}}}(m^2) \in C^\infty$, $\forall n_{u,w,\tilde{u}}$ (including at $m = 0$)

$\Longrightarrow$ susceptibilities with even (odd) no. of S, P are C^∞ ($m \cdot C^\infty$)

If symmetry remains unbroken when probing with external fermion fields
$\Longrightarrow \rho(\lambda; m) \in C^\infty(m^2)$

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Spectrum of GW Dirac operator

Ginsparg-Wilson relation [Ginsparg, Wilson (1982)]: $\{\gamma_5, D\} = 2DR\gamma_5 D$

Restrict to $R = \frac{1}{2}, D^\dagger = \gamma_5 D \gamma_5 \Longrightarrow$ GW relation: $D + D^\dagger = DD^\dagger = D^\dagger D$

domain-wall [Kaplan (1992)]
overlap [Neuberger (1997)]

D normal, spectrum on a circle

$$D\psi_n = \mu_n \psi_n \qquad D(\gamma_5\psi_n) = \mu_n^*(\gamma_5\psi_n) \qquad |1-\mu_n|^2 = 1$$

- pairs of complex conjugate modes $\mu_n \neq \mu_n^*$, spectral density

$$\rho(\lambda; m) = \frac{\mathrm{T}}{\mathrm{V}} \left\langle \textstyle\sum_n \delta(\lambda - \lambda_n) \right\rangle \qquad \lambda_n = |\mu_n| \,\mathrm{sgn}(\mathrm{Im}\mu_n) \quad |\lambda_n| \in (0, 2)$$

- $N_\pm$ chiral zero-modes $\mu_n = 0$, $N'_\pm$ chiral "doubler" modes $\mu_n = 2$

Topological charge $Q = N_+ - N_- = N'_- - N'_+ = -\frac{1}{2}\mathrm{tr}\, \gamma_5 D$

Generating function from the Dirac spectrum

$$\mathcal{W}(\mathrm{J}_{\mathcal{S}}, \mathrm{J}_{\mathcal{P}}; m) - \mathcal{W}(0,0;m)$$
$$= \sum_{\vec{n}\neq 0} \frac{X_0^{n_1} X_0^{*n_2}}{n_1! n_2! n_3!} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \underbrace{s(n_1,k_1)s(n_2,k_2)}_{\text{Stirling numbers of the first kind}} I^{(n_3)}_{N_+^{k_1} N_-^{k_2}}[X,\ldots,X]$$

$$X_0(\mathrm{J}_{\mathcal{S}}, \mathrm{J}_{\mathcal{P}}; m) \equiv (u - w + i\tilde{u})/m^2$$
$$X(\mathrm{J}_{\mathcal{S}}, \mathrm{J}_{\mathcal{P}}; \lambda; m) \equiv 2\left(f(\lambda;m)u + \tilde{f}(\lambda;m)w\right) + f(\lambda;m)^2\left((u-w)^2 + \tilde{u}^2\right)$$
$$f(\lambda;m) \equiv \frac{h(\lambda)}{\lambda^2 + m^2 h(\lambda)} \qquad h(\lambda) = 1 - \tfrac{\lambda^2}{4} \qquad \tilde{f} \equiv f - 2m^2 f^2$$

$$I^{(k)}_{N_+^{k_1} N_-^{k_2}}[g_1,\ldots,g_k] \equiv \left[\prod_{i=1}^{k} \int_0^2 d\lambda_i\, g_i(\lambda_i)\right] \underbrace{\rho^{(k)}_{N_+^{k_1} N_-^{k_2}\, c}(\lambda_1,\ldots,\lambda_k;m)}_{\text{connected eigenvalue correlation function}}$$

Density of zero modes $\lim_{\mathrm{V}\to\infty} \mathrm{V}^{-1}\langle N_+ + N_-\rangle = 0$ in the thermo limit, but higher cumulants, correlations with complex modes do not vanish

First-order constraints

To lowest order, since $CP \Rightarrow \mathcal{W}(-\tilde{u}) = \mathcal{W}(\tilde{u})$

$$\mathcal{W} = \mathcal{W}|_{j=0} + \frac{\chi_\pi}{2} u + \frac{\chi_\delta}{2} w + \frac{1}{2m^2}\left(\frac{\chi_\pi - \chi_\delta}{4} - \frac{\chi_t}{m^2}\right)\tilde{u}^2 + \dots$$

$|\chi_\delta| \le \chi_\pi$, requirements from chiral symmetry restoration reduce to

$$\lim_{m\to 0} \frac{\chi_\pi}{4} = \lim_{m\to 0} I^{(1)}[f] = \lim_{m\to 0}\left[\int_0^2 d\lambda \, \frac{\rho(\lambda; m)}{\lambda^2 + m^2} + o(m)\right] < \infty$$

$$\frac{\chi_\pi - \chi_\delta}{4} = 2m^2 I^{(1)}[f^2] = \frac{\chi_t}{m^2} + O(m^2)$$

$\mathrm{U}(1)_A$ order parameter

$$\frac{\Delta}{2} \equiv \lim_{m\to 0} \frac{\chi_\pi - \chi_\delta}{8} = \lim_{\epsilon\to 0}\lim_{m\to 0}\int_0^\epsilon d\lambda \, \frac{m^2\rho(\lambda; m)}{(\lambda^2 + m^2)^2} = \lim_{m\to 0}\frac{\chi_t}{m^2} < \infty$$

Higher-order coefficients involve higher-point eigenvalue correlators

Spectral density and $\mathrm{U}(1)_A$ breaking I

$\mathrm{SU}(2)_A$ restoration and effective $\mathrm{U}(1)_A$ breaking generally compatible, but $\mathrm{U}(1)_A$ breaking requires some kind of singular behaviour of ρ

$$\infty > \lim_{m\to 0} \frac{\chi_\pi}{4} = \lim_{m\to 0} \int_0^2 d\lambda \, \frac{\rho(\lambda; m)}{\lambda^2 + m^2} \geq \lim_{m\to 0} \int_0^\epsilon d\lambda \, \frac{\rho(\lambda; m)}{\lambda^2 + m^2}$$
$$\geq \lim_{\eta\to 0} \lim_{m\to 0} \int_\eta^\epsilon d\lambda \, \frac{\rho(\lambda; m)}{\lambda^2 + m^2} = \lim_{\eta\to 0} \int_\eta^\epsilon d\lambda \, \frac{\rho(\lambda; 0)}{\lambda^2} \Rightarrow \frac{\rho(\lambda; 0)}{\lambda^2} \text{ integrable}$$

$\Longrightarrow \rho(\lambda; m) = \sum_{n=0}^\infty \rho_n(m^2)|\lambda|^n$, $\rho_n \in C^\infty$ cannot give $\Delta \neq 0$

[Aoki, Fukaya, Taniguchi (2012), Kanazawa, Yamamoto (2016)]

$$\rho(\lambda; 0) = O(\lambda^2) \Longrightarrow \rho_0(m^2), \rho_1(m^2) = O(m^2)$$

$$\frac{\Delta}{2} = \lim_{\epsilon\to 0} \lim_{m\to 0} \int_0^\epsilon d\lambda \left[\left(\frac{m^2}{\lambda^2 + m^2} \right)^2 \left(\frac{\rho_0(m^2)}{m^2} + \frac{\rho_1(m^2)}{m^2}\lambda \right) + \frac{m^2\lambda^2}{(\lambda^2 + m^2)^2} O(\lambda^0) \right]$$
$$\leq \lim_{\epsilon\to 0} \lim_{m\to 0} \int_0^\epsilon d\lambda \left[\frac{\rho_0(m^2)}{m^2} + \frac{|\rho_1(m^2)|}{m^2}\lambda \right] \leq \lim_{\epsilon\to 0} \int_0^\epsilon d\lambda \, \left[\rho_0'(0) + |\rho_1'(0)|\lambda \right] = 0$$

Spectral density and $\mathrm{U}(1)_A$ breaking II

$$\rho(\lambda; m) = \rho(\lambda; 0) + m^2 \frac{\rho(\lambda;m)-\rho(\lambda;0)}{m^2} = \rho(\lambda; 0) + m^2 \rho_1(\lambda; m)$$

$$\rho_1(\lambda; m) = \rho_{1+}(\lambda; m) - \rho_{1-}(\lambda; m) \qquad \rho_{1\pm}(\lambda; m) = \pm\frac{\rho_1(\lambda;m)\pm|\rho_1(\lambda;m)|}{2}$$

Positivity of ρ implies $m^2\rho_{1-}(\lambda; m) \leq \rho(\lambda; 0)$, can drop both

$$\frac{\Delta}{2} = \lim_{\epsilon\to 0}\lim_{m\to 0}\int_0^{\epsilon} d\lambda \left(\frac{m^2}{\lambda^2+m^2}\right)^2 \rho_{1+}(\lambda; m)$$

$$\leq \lim_{\epsilon\to 0}\lim_{m\to 0}\int_0^{\epsilon} d\lambda\, \rho_{1+}(\lambda; m) = \lim_{\epsilon\to 0}\int_0^{\epsilon} d\lambda\, \rho_{1+}(\lambda; 0)$$

$\Delta \neq 0$ requires one of the following:

- $\lim_{m\to 0}\rho_{1+}(\lambda; m) = \infty$ in a set of nonzero measure, $\rho(\lambda; m) \notin C^\infty(m^2)$
- if $\rho_{1+}(\lambda; m)$ bounded in λ for $m \neq 0$, the bound must diverge as $m \to 0$
- if $\rho_{1+}(\lambda; m)$ singular at $\lambda = 0$ for $m \neq 0$, the singularity must become not integrable as $m \to 0$ (no singularities expected elsewhere)

If $\rho(\lambda; m) \in C^\infty(m^2)$, $\rho_{1\pm}(\lambda; 0)$ both well defined, necessary condition: $\rho_{1+}(\lambda; 0)$ contains $\delta(\lambda)$

Spectral density and $\mathrm{U}(1)_A$ breaking III

For arbitrary $\zeta \in \mathbb{R}^+$

$$\infty > \lim_{m\to 0} \frac{\chi_\pi}{4} \geq \lim_{m\to 0} \int_0^{\zeta m} d\lambda \, \frac{m^2}{\lambda^2 + m^2} \rho_{1+}(\lambda; m)$$
$$= \lim_{m\to 0} \int_0^{\zeta} dz \, \frac{m\rho_{1+}(mz; m)}{z^2+1} = \int_0^{\zeta} dz \, \frac{\bar{\rho}_{1+}(z)}{z^2+1}$$

$\bar{\rho}_{1+}(z) < \infty$, must grow more slowly than z, but could be $\equiv 0$

$$\frac{\Delta}{2} \geq \int_0^{\zeta} dz \, \frac{\bar{\rho}_{1+}(z)}{(z^2+1)^2}$$

Sufficient condition: if $\bar{\rho}_{1+}(z)$ is a nonzero measure then $\Delta \neq 0$

Examples:

$$\rho_{1+}(\lambda; m) = \frac{A}{\pi} \frac{\Lambda m^2}{(\lambda\Lambda)^2 + m^4} \qquad \rho_{1+}(0; m) \underset{m\to 0}{\to} \infty \qquad \bar{\rho}_{1+}(z) = A\delta(z)$$

$$\rho_{1+}(\lambda; m) = A m^{1-\alpha} \lambda^{\alpha}\,, \quad |\alpha| < 1 \qquad \rho_{1+}(\lambda; 0^+) = \infty \qquad \bar{\rho}_{1+}(z) = A z^{\alpha}$$

Spectral density and $\mathrm{U}(1)_A$ breaking: Singular peak I

$\mathrm{U}(1)_A$-breaking singular peak compatible with χSR and $\rho \in C^\infty(m^2)$:

$$\rho(\lambda; m) \underset{\lambda\to 0}{\simeq} \rho_{\mathrm{peak}}(\lambda; m) = m^2 \rho_{1+}(\lambda; m)$$

$$\rho_{\mathrm{peak}}(\lambda; m) = \frac{\Delta}{2} \frac{m^2 \gamma(m^2)}{\lambda^{1-\gamma(m^2)}} \qquad \gamma > 0\,,\ \gamma \in C^\infty\,,\ \gamma = O(m^2)$$

- Singularity becomes nonintegrable in the chiral limit, $\bar{\rho}_{1+}(z) = \Delta\delta(z)$
- Peak modes must be closely related to topology

$$\lim_{m\to 0} \frac{n_{\mathrm{peak}}}{m^2} = \lim_{m\to 0} \frac{2}{m^2} \int_0^2 d\lambda\, \rho_{\mathrm{peak}}(\lambda; m) = \Delta = \lim_{m\to 0} \frac{\chi_t}{m^2}$$

- Compatible with commutativity of thermo and chiral limits!

$$\lim_{m\to 0} |m|^{-1} \rho_{\mathrm{peak}}(|m| z; m) = \Delta\delta(z) \qquad z \in [-1, 1]$$

Necessary condition for commutativity [Azcoiti (2023)]

- Same for $\rho_{\mathrm{peak}}(\lambda; m) = \frac{m^2\gamma(m^2)}{|\lambda|} \phi\left(\gamma(m^2) \ln \frac{2}{|\lambda|}\right)$, $\int_0^\infty dx\, \phi(x) = \frac{\Delta}{2}$

Spectral density and $\mathrm{U}(1)_A$ breaking: Singular peak II

- Singularity must become nonintegrable while $n_{\mathrm{peak}} \simeq \chi_t \to 0$, expect similar behaviour as $m \to 0$ or $\mathrm{T} \to \infty$
- To see the peak one needs $n_{\mathrm{peak}}(\mathrm{V}, \mathrm{T}, m) \simeq \frac{\mathrm{V}\chi_t}{\mathrm{T}} \sim 1$, requires large V

$$\mathrm{V} \gtrsim \frac{\mathrm{T}}{\chi_t} = O(T^b, m^{-2})$$

- Existence of the peak at $m \neq 0$ quite well established, scaling with m yet unknown

  [Edwards *et al.* (1999), Cossu *et al.* (2013), Alexandru, Horváth (2015), Dick *et al.* (2015), Brandt *et al.* (2016), Tomiya *et al.* (2017), HotQCD (2019), JLQCD (2021), Vig, Kovács (2021), Kaczmarek *et al.* (2021), Meng *et al.* (2023), Alexandru *et al.* (2024)]

Second-order constraints – two-point function

More constraints from second-order coefficients in the expansion of $\mathcal{W}$
Two constraints involving the two-point function

$$\rho_c^{(2)}(\lambda, \lambda'; m) = \lim_{\mathrm{V}\to\infty} \frac{\mathrm{T}}{\mathrm{V}} \Big(\langle \textstyle\sum_{n\neq n'} \delta(\lambda - \lambda_n)\delta(\lambda' - \lambda_{n'})\rangle \\ - \langle \textstyle\sum_n \delta(\lambda - \lambda_n)\rangle\langle \textstyle\sum_{n'} \delta(\lambda' - \lambda_{n'})\rangle \Big)$$

$$4m^2 I^{(2)}[f, f] = -\left(\lim_{\mathrm{V}\to\infty} \frac{\mathrm{T}}{\mathrm{V}} \frac{\langle N_0 \rangle^2}{m^2} \right) + O(m^2)$$
$$I^{(2)}[\hat{f}, \hat{f}] = O(m^0)$$

$$I^{(2)}[g_1, g_2] \equiv \int_0^2 d\lambda \int_0^2 d\lambda' \, g_1(\lambda) g_2(\lambda') \rho_c^{(2)}(\lambda, \lambda'; m) \qquad \hat{f} \equiv f - m^2 f^2$$

Assume $\rho_c^{(2)}$ ordinary function (no δs)

Two-point function finite at the origin

If $\rho_c^{(2)}(\lambda, \lambda'; m)$ finite at the origin

$$\rho_c^{(2)}(\lambda, \lambda'; m) = A(m) + B(\lambda, \lambda'; m)$$

$|B(\lambda, \lambda'; m)| \leq b(\lambda^2 + \lambda'^2)^{\frac{\beta}{2}}$ with $\beta < 1$

$$\lim_{m\to 0} 4m^2 I^{(2)}[f, f] = \pi^2 A(0) = -\mathrm{T} \lim_{m\to 0} \lim_{\mathrm{V}\to\infty} \left\langle \frac{N_0}{m\sqrt{\mathrm{V}}} \right\rangle^2$$

$$\lim_{m\to 0} (4m)^2 I^{(2)}[\hat{f}, \hat{f}] = \pi^2 A(0) = 0$$

$\Rightarrow$ measure of $\frac{N_0}{m\sqrt{\mathrm{V}}}$ concentrated in zero in thermo and chiral limit

$\Rightarrow \lim_{m\to 0} \frac{\chi_t}{m^2} = 0$

$\Rightarrow \Delta = 0$

No $\mathrm{U}(1)_A$-breaking topological effects if $\rho_c^{(2)}(\lambda, \lambda'; m)$ finite at zero

Localised near-zero modes

$T > T_c$ modes localised below "mobility edge" $\lambda_c \Rightarrow$ Poisson statistics

[Review: MG, Kovács (2021)]

If near-zero modes are localised, expect

$$|\rho_c^{(2)}(\lambda,\lambda';m)| \leq C\rho(\lambda;m)\rho(\lambda';m) \quad \text{if } \lambda,\lambda' < \lambda_c \text{ or } \lambda < \lambda_c < \lambda'$$

- purely Poisson spectrum: $\rho_c^{(2)}(\lambda,\lambda') \propto \rho(\lambda)\rho(\lambda')$ [Kanazawa, Yamamoto (2016)]
- $\rho_c^{(2)}(\lambda_{\rm loc},\lambda_{\rm deloc})/[\rho(\lambda_{\rm loc})\rho(\lambda_{\rm deloc})] \sim O(1)$ and negative

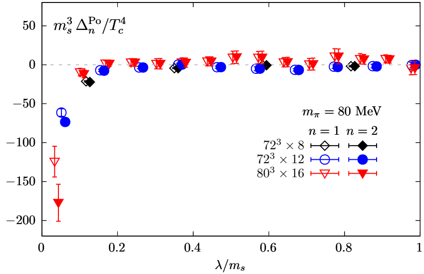
suffices $|\rho_c^{(2)}(\lambda_{\rm loc},\lambda_{\rm deloc})| \leq C'\rho(\lambda_{\rm loc})$

$$\text{If } \lambda_c \underset{m\to 0}{\not\to} 0 \Rightarrow \mathrm{T} \lim_{m\to 0}\lim_{\mathrm{V}\to\infty} \left\langle \frac{N_0}{m\sqrt{\mathrm{V}}} \right\rangle^2 = -\lim_{m\to 0} 4m^2 I^{(2)}[f,f] = 0$$

$\mathrm{U}(1)_A$-breaking topological effects require

- either localised modes disappear, $\lambda_c \to 0$ as $m \to 0$
- or another mobility edge $0 < \lambda_c' < \lambda_c$ close to zero

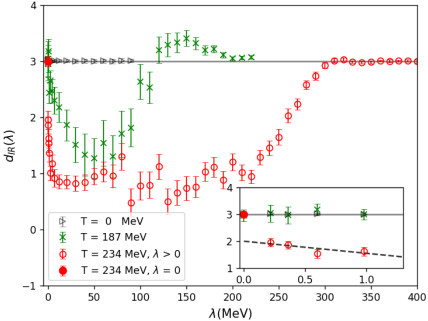
Correlations and localisation of near-zero modes



Non-Poissonian repulsion of near-zero modes

$$\Delta_n^{\rm Po} = m^{n-2}\left[\partial_m^n \rho - (\partial_m^n \rho)^{\rm Poisson}\right]$$

[HotQCD (2019)]



Near-zero modes delocalised, second mobility edge?

$$\mathcal{N}_*[\psi_n] = \sum_x \min(\mathrm{L}^3/\mathrm{T}\|\psi_n(x)\|^2, 1)$$

$$\langle \sum_n \mathcal{N}_*[\psi_n]\delta(\lambda - \lambda_n)\rangle \sim \mathrm{L}^{d_{\rm IR}(\lambda)}$$

[Meng *et al.* (2023)]

$\mathrm{U}(1)_A$ breaking and topological charge distribution

Using (anomalous) symmetry properties of $\mathcal{Z}$

$$F(\theta; m) = -\lim_{\mathrm{V}\to\infty} \frac{\mathrm{T}}{\mathrm{V}} \ln \mathcal{Z}(\theta; m) = -\lim_{\mathrm{V}\to\infty} \hat{\mathcal{W}}(m^2 + u_\theta, w_\theta, \tilde{u}_\theta)$$

$$u_\theta = -m^2 \left(\sin\tfrac{\theta}{2}\right)^2 \qquad w_\theta = m^2 \left(\sin\tfrac{\theta}{2}\right)^2 \qquad \tilde{u}_\theta = m^2 \sin\theta$$

By CP, $\hat{\mathcal{W}}$ depends only on $\tilde{u}_\theta^2 = 4w_\theta(m^2 - w_\theta)$

In the symmetric phase F can be expanded in powers of $w_\theta = m^2 \left(\sin\frac{\theta}{2}\right)^2$, coefficients $\sum_{i=1}^{N}(m^2)^{p_i}\mathcal{A}_{n_i}(m^2)$ can be expanded in powers of m^2

$$F(\theta; m) - F(0; m) = (1-\cos\theta)m^2\Delta + O(m^4) = (1-\cos\theta)\chi_t + O(m^4)$$

Same as a free instanton gas if $\Delta \neq 0$

With effective methods: [Kanazawa, Yamamoto (2015)]

Singular peak from topological fluctuations

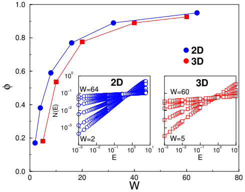
Peak reproduced in weakly interacting, dilute instanton gas model: peak modes from mixing of instanton zero modes [Kovács (2023)]

- interaction via fermion determinant $\rightarrow$ instanton-antiinstanton molecules + free instanton gas component
- density of peak modes n_{peak} matches density of free instanton gas

$$n_{\mathrm{peak}} \approx n_{\mathrm{inst}} = \chi_t \propto m^2 \Rightarrow \Delta \neq 0$$

- m-dependent power α, peak "height" both decreasing as $m \rightarrow 0$
- in a similar cond-mat model of a disordered system w/ chiral symmetry
  - $\alpha \rightarrow -1$ as disorder ($\sim 1/n_{\mathrm{inst}}$) increases [Evangelou, Katsanos (2003)]
  - mobility edge near zero [García-García, Cuevas (2006)]

$\longrightarrow$



$\Rightarrow$ expected to be reproduced in the instanton model

Summary and outlook

- $\mathrm{U}(1)_A$ breaking compatible with χSR but requires
  1. singular spectral density $\rho \sim \frac{O(m^4)}{\lambda}$ (or some other singular behaviour)
  2. singular two-point function, and near-zero modes *not* localised
  3. ideal instanton gas-like topology
- Required spectral features occur naturally if gauge field configurations at small m include weakly interacting topological objects of unit charge

Open issues:

- check expected properties of instanton model
- other sectors, larger N_f
- test against numerical results

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