QCD Anderson transition with overlap valence quarks and influence of external magnetic fields

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Based on: PRD **109** 074512, PoS **LATTICE2024** 189, PoS **LATTICE2025** 123

QCD Anderson transition ...

1. ... with overlap valence quarks (on a twisted-mass sea)

2. ... and influence of external magnetic fields

3. Algebraic connections of overlap eigenmodes

Motivation

Fundamental transitions in QCD

- Chiral restoration
- Deconfinement

Open question

Is there a relation between both transitions?

- QCD Anderson transition related to both phenomena
- In this work: Focus on relation to chiral restoration

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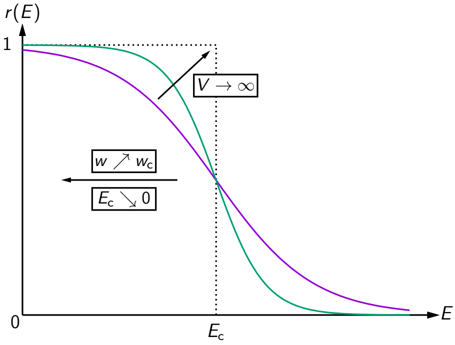
Anderson transition in condensed matter physics

[P. W. Anderson, PR **109** (1958) 1492], [F. Evers, A. D. Mirlin, RMP **80** (2008) 1355]

- Describes metal-insulator transition in disordered solids
- In metal phase low-lying eigenmodes of Hamiltonian delocalized
  $\Rightarrow$ Conductivity
- Above critical disorder all eigenmodes localized
  $\Rightarrow$ No conductivity

Anderson transition

- Delocalized modes separated from localized modes by energy threshold E_c (*mobility edge*)
- Above critical disorder strength w_c all modes localized



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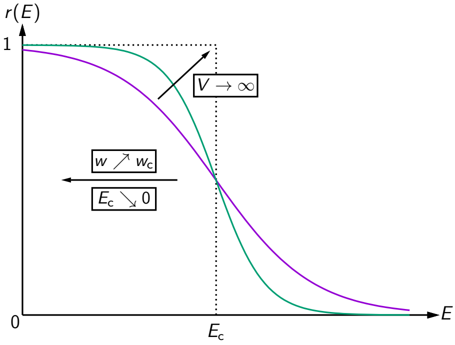

Analogous transition in QCD

[M. Giordano, T. G. Kovács, Universe **7** (2021) 194]

- Hamilton operator
  ↳ Dirac operator
- Disorder strength
  ↳ Temperature T
- Low-lying modes localized
- Higher ones delocalized
- Below T_0 all modes delocalized (no mobility edge)

Relation to . . .

. . . (de)confinement

- Eigenmodes tend to localize in sinks of Polyakov loop

  [L. Holicki, E.-M. Ilgenfritz, L. von Smekal, PoS **LATTICE2018** (2019) 180]

- Quenched QCD: T_0 coincides with deconfining phase transition

  [T. G. Kovács, R. Á. Vig, PRD **97** (2018) 014502]

Relation to ...

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... chiral symmetry restoration/breaking

- Some studies suggest $T_0 = T_{\mathrm{pc}}$ (pseudocritical temperature of chiral crossover, pion mass $m_\pi \neq 0$)
- No Goldstone bosons in chiral limit, if near-zero modes localized
  [M. Giordano, JHEP **12** (2022) 103]
  $\Rightarrow T_0 \geq T_{\mathrm{c}}$ (temperature of chiral phase transition, $m_\pi \to 0$)
- Near-zero modes produce chiral condensate (Banks-Casher relation)
  [T. Banks, A. Casher, NPB **169** (1980) 103–125]

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Mixed action setup

Chiral lattice fermions

- Compute low-lying eigenmodes of overlap operator:

$$D_{\text{ov}} = \frac{\rho}{a}\left(1 + \operatorname{sgn} K\right)$$

- Wilson kernel: $K = aD_{\text{W}} - \rho$
- Optimize locality with parameter $\rho \in (0, 2)$

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Gauge configurations

- From *twisted mass at finite temperature* (tmfT) collaboration
  [F. Burger, E.-M. Ilgenfritz, M. P. Lombardo, A. Trunin, PRD **98** (2018) 094501]
  - Twisted mass Wilson fermions, Iwasaki gauge action
  - $N_{\text{f}} = 2 + 1 + 1$: two degenerate light, physical strange & charm quarks
  - Lattice spacing a from [N. Carrasco et al., NPB **887** (2014) 19–68]
  - m_π from [M. Werner et al., EPJA **56** (2020) 61]
  - T_{pc} from [A. Y. Kotov, M. P. Lombardo, A. Trunin, PLB **823** (2021) 136749]

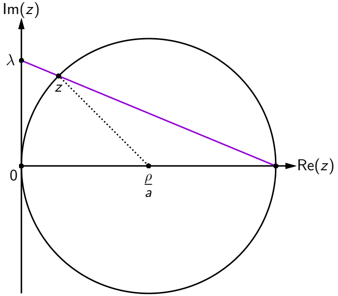
Observable and configurations

Localization measure

- Relative volume of eigenmode:

$$r(\lambda) = \frac{P^{-1}(\lambda)}{N_s^3 N_t} \in [0, 1]$$

- $P(\lambda) = \sum_n \left[v_\lambda(n)^\dagger v_\lambda(n)\right]^2$



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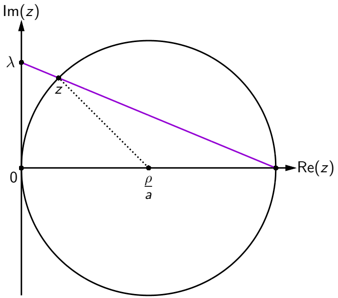


Set of ensembles	N_t	T/T_{pc}
D210 $N_s = 48$ $a = 0.0619(18)$ fm $L = 2.97(9)$ fm $m_\pi = 225(7)$ MeV $T_{pc} = 171(6)$ MeV	4	4.66(23)
	6	3.11(16)
	8	2.33(12)
	10	1.86(9)
	12	1.55(8)
	14	1.33(7)
	16	1.17(6)
	18	1.04(5)
	20	0.93(5)
	24	0.78(4)

- For $N_t = 18$:

$$T = 177(5)\,\text{MeV} \approx T_{pc}$$

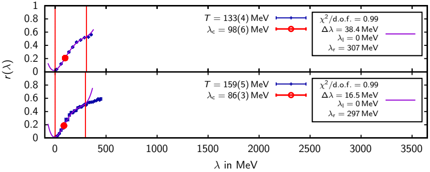
- For $N_t = 24$:

$$T = 133(4)\,\text{MeV} \approx T_c$$

Relative volume for each T

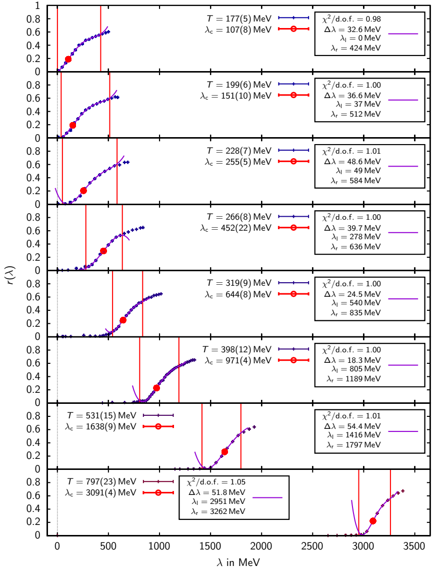
$$T_c = 132^{+3}_{-6}\,\text{MeV}$$

[H.-T. Ding et al., PRL **123** (2019) 062002]



$$T_{pc} = 171(6)\,\text{MeV}$$

- Inflection point λ_c as estimate for mobility edge
- No expected vanishing in the range $[T_c, T_{pc}]$



Temperature dependence

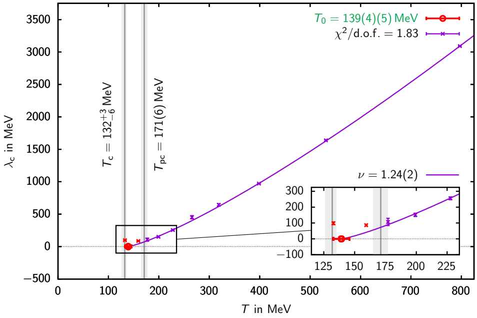
- Exclude lowest two temperatures
- λ_c seems to approach a constant here

- Zero coincides with T_c
- $\nu \approx 1.44$ for unitary Anderson model

  [L. Ujfalusi, I. Varga, PRB **91** (2015) 184206]

Scaling fit

$$\lambda_c(T) = b\,(T - T_0)^{\nu}$$



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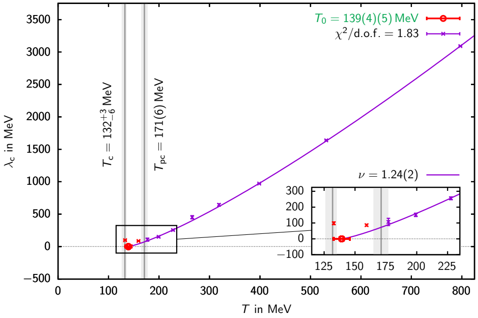
Key question

Why does the data below T_{pc} deviate from the fit?

Scaling fit

$$\lambda_c(T) = b\,(T - T_0)^{\nu}$$



Possible scenario

- If modes with low $r(\lambda)$ scale with $L^{d_{\mathrm{IR}}}$, where $d_{\mathrm{IR}} \in (0, 3]$ $\Rightarrow$ Not localized
- Determine d_{IR} [A. Alexandru, I. Horváth, PRL **127** (2021) 052303] $\Rightarrow$ 2nd mobility edge $\lambda_{\mathrm{IR}} = 0$ for $T > T_{\mathrm{IR}} \in [200, 250]$ MeV
- Modes below λ_{IR} delocalized, higher ones localized

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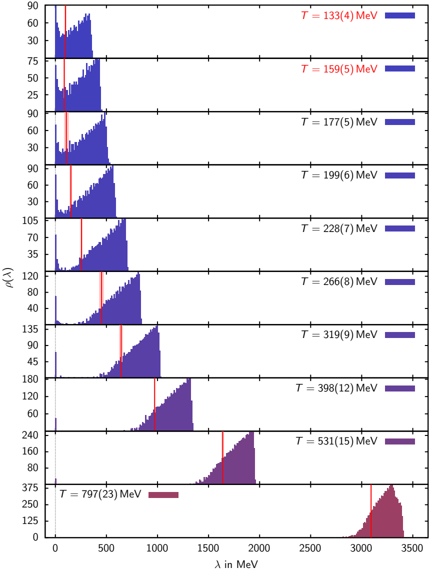
- Decrease $T \Rightarrow \lambda_{\mathrm{IR}}$ might increase and annihilate λ_c
- Caution when λ_c hits infrared spectrum at $\approx T_{pc}$



Level spacing statistics

- Investigate level spacing ratios:

$$\tilde{r}_n = \min\left(\frac{s_{n+1}}{s_n}, \frac{s_n}{s_{n+1}}\right) \in [0, 1]$$

[R. Shanker, H. Pandey, S. Sharma, JSPC **4** (2025) 100224]

- $s_n = \lambda_{n+1} - \lambda_n$
- $\langle \tilde{r} \rangle \approx 0.603$ for Gaussian unitary ensemble (delocalized)
- $\langle \tilde{r} \rangle \approx 0.386$ for Poisson distribution (localized)

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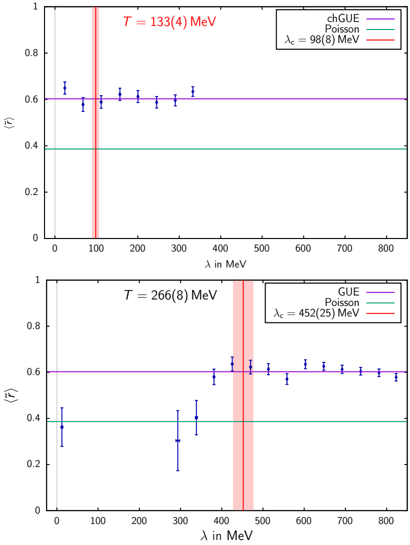
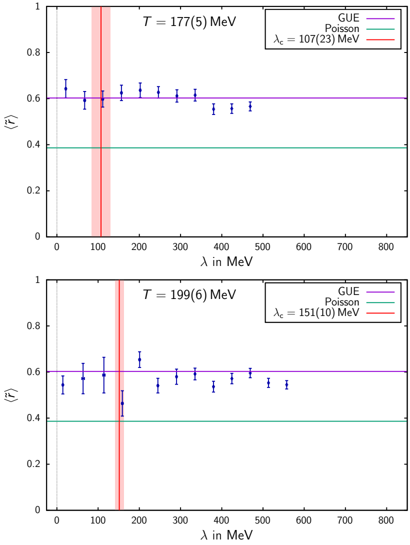
- According to $\langle \tilde{r} \rangle$ no localization at $T \approx T_c$ but for $T > T_{IR}$



Onset of localization?

- No localization at $T \approx T_{pc}$
- Slight indication of a dip at $T \approx 200\,\text{MeV}$
- Seems to happen in the range $[T_{pc}, T_{IR}]$
- So far agreement with d_{IR} study
  [X.-L. Meng et al., JHEP **12** (2024) 101]

- Improved statistics and smaller binsize required
- Larger volume & physical pion desirable (currently $L \approx 3\,\text{fm}$, $m_\pi \approx 225\,\text{MeV}$)
- Reduce computational costs



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Staggered fermions in magnetic fields

Staggered operator

$$D_{\text{st}}^{f}(n,l) = \sum_{\mu} \frac{\eta_\mu(n)}{2a}\left[U_\mu(n)u_\mu^f(n)\delta_{n+\hat{\mu},l} - U_\mu^\dagger(n-\hat{\mu})u_\mu^{f*}(n-\hat{\mu})\delta_{n-\hat{\mu},l}\right]$$

- Staggered phases:
  $\eta_1(n) = 1\,,\ \eta_\mu(n) = (-1)^{\sum_{\nu<\mu} n_\nu}$
- Flavor dependent $u_\mu^f(n) \in \mathrm{U}(1)$ implement external magnetic field

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Quantization condition on torus

- Magnetic flux ambiguous:
  $\Phi = q_f BS \quad \vee \quad \Phi_{\mathrm{c}} = -q_f B(L^2 - S)$
- With $e^{\mathrm{i}\Phi} \stackrel{!}{=} e^{\mathrm{i}\Phi_{\mathrm{c}}}$ and $q_{\mathrm{d}} = -e/3$:
  $\Rightarrow \quad eB = 6\pi N_{\mathrm{b}}/L^2\,, \quad N_{\mathrm{b}} \in \mathbb{Z}$

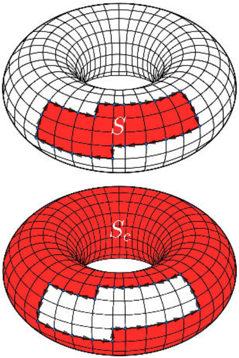

Figure: From A. D. M. Valois

Magnetic fields in QCD

Motivation

Magnetic fields reduce chiral & deconfinement transition temperature

[G. S. Bali et al., JHEP **02** (2012) 044]

⇒ Expect similar behavior for Anderson transition temperature T_0

(Inverse) magnetic catalysis

- Magnetic field enhances chiral condensate away from T_{pc}
- Condensate gets suppressed in region close to T_{pc} [F. Bruckmann, G. Endrődi, T. G. Kovács, JHEP **04** (2013) 112]

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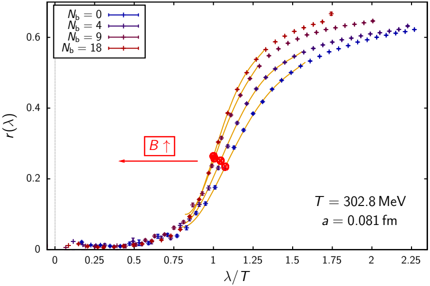
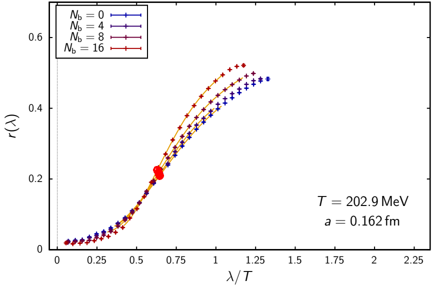
Setup

- Compute 400 eigenmodes of staggered operator
- $2+1$ flavors of unimproved staggered fermions, tree-level Symanzik-improved gauge action
- $N_{\mathrm{s}} = 24,\ N_{\mathrm{t}} \in \{6, 8\}$
- Vary T via β

Relative volume: $N_s = 24$, $N_t = 6$





- B decreases λ_c at high T
- Seems to stay constant when Banks-Casher gap closes

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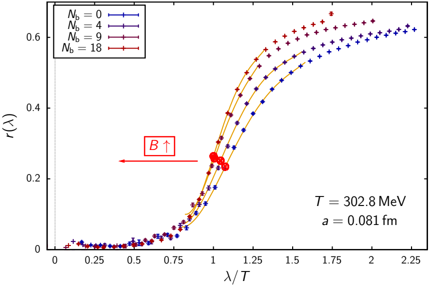

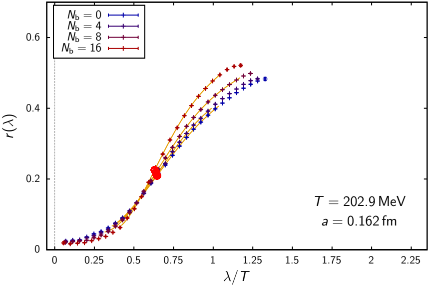

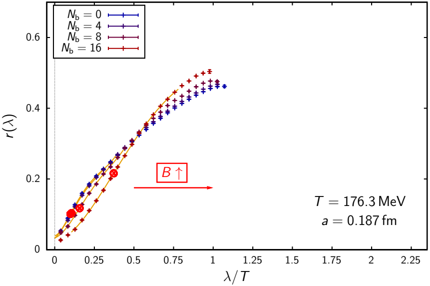

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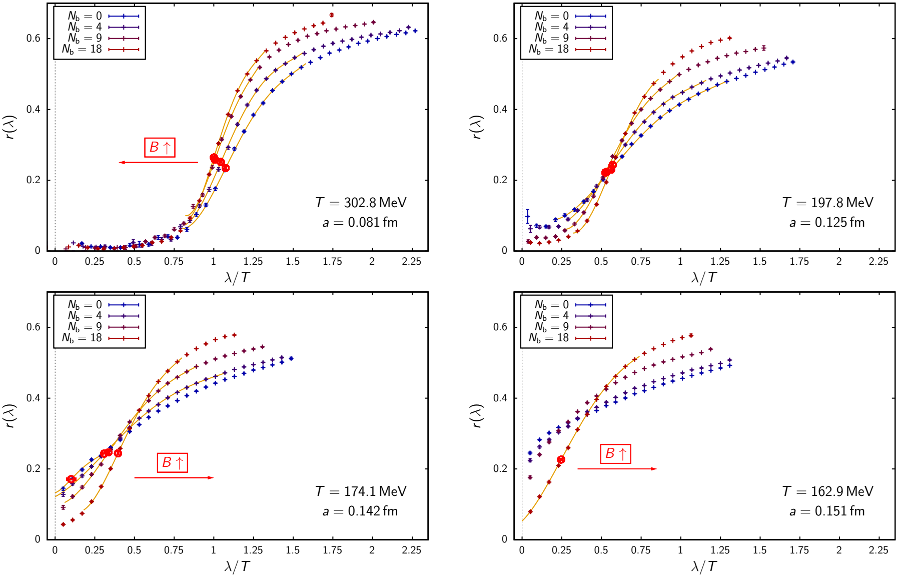
Turn around

For lower T mobility edge gets increased by $B \Rightarrow T_0$ reduced

Relative volume: $N_s = 24$, $N_t = 8$



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Eigenpairs of D_{ov} from $D_{\text{ov}}^\dagger D_{\text{ov}}$

- $Dv = \lambda v \quad \Leftrightarrow \quad D(\gamma_5 v) = \lambda^*(\gamma_5 v)$
  $\Rightarrow$ Sufficient to compute just upper (or lower) semicircle
- Hermitian operators numerically more robust and cheaper
- γ_5-hermiticity: $(\gamma_5 D)^\dagger = \gamma_5 D := H \quad \Rightarrow \quad D^\dagger D = H^2$ hermitian

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- v and $\gamma_5 v$ also eigenvectors of H^2 with degenerate eigenvalues $|\lambda|^2$
- $[H^2, \gamma_5] = 0 \quad \Rightarrow$ Can choose chiral basis $w_{\text{L/R}}$ of this eigenspace
- Get rid of degeneracy by restricting to, e.g., right-handed subspace:
  $D^\dagger D = 2 + (1 + \gamma_5)\,\text{sgn}\,K$
- Reconstruction:
  $\lambda_\pm = \frac{1}{2}\left(|\lambda|^2 \pm \mathrm{i}|\lambda|\sqrt{4 - |\lambda|^2}\right) \quad \Rightarrow \quad v \propto (Hw_{\text{R}} - \lambda_\pm w_{\text{R}})$

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- However condition number gets worse: $\kappa(D^{\dagger} D) = \kappa(D)^{2}$

Eigenpairs of D_{ov} from $H = \gamma_5 D_{\text{ov}}$

- Condition number: $\kappa(\gamma_5 D) = \kappa(D)$
- $Hu_\pm = \pm|\lambda| u_\pm$ with non-chiral $u_\pm$ due to $[H, \gamma_5] \neq 0$
- But $w_{\text{L/R}} = P_{\text{L/R}}\, u_\pm \quad \Rightarrow$ Formulas can still be applied
- Just positive (or negative) part of spectrum sufficient

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How to get rid of redundant part of spectrum?

- Simple chiral projection does not work here
- Reverse direction: $u_\pm \propto (v \pm \frac{\lambda}{|\lambda|}\gamma_5 v)$
- Idea: Filter initial vector u_0 for Krylov solvers
- Want something acting like $\Theta(H) = (1 + \text{sgn}(H))/2$
  - Approximate with Chebyshev polynomials
  - Amplify positive and suppress negative modes (e.g. also with Chebyshev polynomials)

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Thank you!

Implementation of the overlap operator

- High-dimensional matrix:

$$d = \underbrace{4}_{\text{Dirac}} \cdot \underbrace{3}_{\text{color}} \cdot \underbrace{\prod_{\mu=1}^{4} N_\mu}_{\text{lattice sites}}$$

- $N_s = 24,\ N_t = 4$: $d \approx 660\,\text{k}$ $\Rightarrow d^2 \approx 440\,\text{billion complex entries} \gtrapprox 3.2\,\text{TB storage (single precision)}$
- $N_s = 48,\ N_t = 24$: $\gtrapprox 7.2\,\text{PB}$
- No chance of storing the whole operator!

- Implement matrix-vector product $D_{\text{ov}} v$ instead
- Rational approximation for inverse square root:

$$\text{sgn}\,K = \frac{K}{\sqrt{(\gamma_5 K)^2}} \approx K \left(\alpha_0 + \sum_{j=1}^{k} \frac{\alpha_j}{(\gamma_5 K)^2 + \beta_j} \right)$$

- Solve system of equations $\forall j$:

$$\left((\gamma_5 K)^2 + \beta_j\right) w_j = v$$

- Conjugate gradient method

Krylov subspace methods

- Due to high dimension entire diagonalization not feasible!
- Compute small part of spectrum using Krylov subspace methods

- p-dimensional Krylov subspace to matrix A and start vector x_1:

$$\mathcal{K}_p(A, x_1) = \text{span}\left\{x_1, Ax_1, A^2x_1, \dots A^{p-1}x_1\right\}$$

- Let $u_1, u_2, \dots u_p$ be an orthonormal basis of the Krylov subspace
- Projection onto subspace:

$$H = U^\dagger A U$$

- Compute eigenvalues θ_i and eigenvectors y_i of H
- Approximate eigenpairs of A:

$$\lambda_i \approx \theta_i \qquad \text{(Ritz values)}$$
$$v_i \approx U y_i \qquad \text{(Ritz vectors)}$$

- Best approximations in $\mathcal{K}_p(A, x_1)$

Krylov-Schur method (SLEPc library)

Arnoldi method

- Successively extend Krylov subspace with $\omega := Au_j$
- Compute matrix elements $H_{ij} = u_i^\dagger \omega$
- Orthogonalize ω to $u_1, u_2, \dots u_j$
- Compute $H_{j+1,j} = \|\omega\|_2$ and set $u_{j+1} = \omega / \|\omega\|_2$

- H takes upper Hessenberg form of dimension p
- Arnoldi decomposition:

$$AU = UH + (Au_p)e_p^\dagger$$

- Krylov decomposition:

$$AU = UB + u_{p+1} b_{p+1}^\dagger$$

- Krylov-Schur: B has 1×1 or 2×2 blocks on the diagonal

Repeat

- Build Krylov decomposition using Arnoldi method
- Transform into Krylov-Schur decomposition
- Reorder blocks according to desired eigenvalues
- Truncate Krylov subspace