The finite baryon density frontier in lattice QCD

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Wuppertal-Budapest collaboration.

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New Trends in Thermal Phases of QCD
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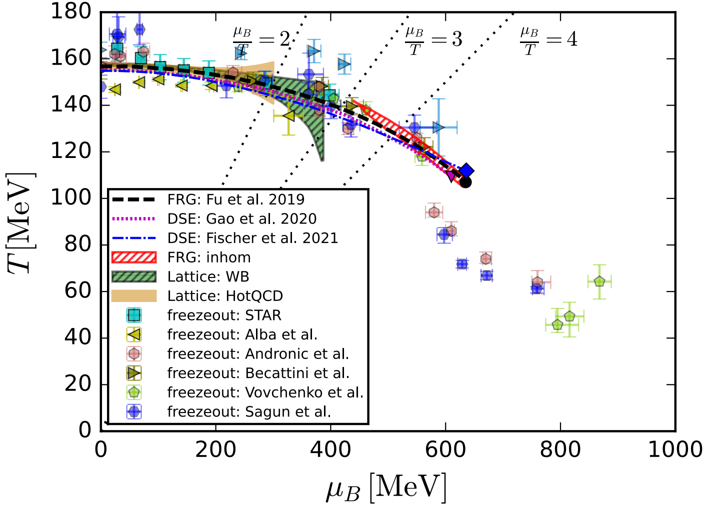
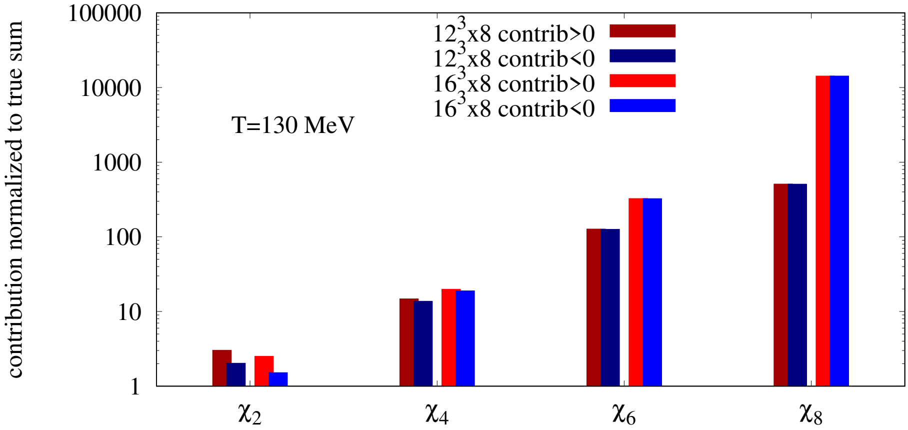
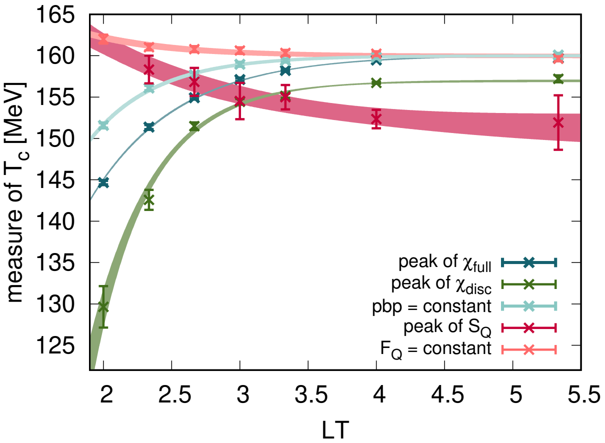


Figure by J. Pawlowski (version 2025) [e.g. 2603.11135]



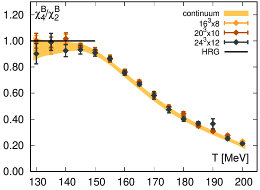
$$\frac{1}{VT^3}\log Z(T,V,\mu_B) = \frac{p_0}{T^4} + \frac{\chi_2}{2!}\left(\frac{\mu_B}{T}\right)^2 + \frac{\chi_4}{4!}\left(\frac{\mu_B}{T}\right)^4 + \frac{\chi_6}{6!}\left(\frac{\mu_B}{T}\right)^6 + \frac{\chi_8}{8!}\left(\frac{\mu_B}{T}\right)^8 + \dots$$

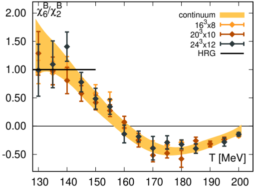
How is physics distorted in a smaller volume?

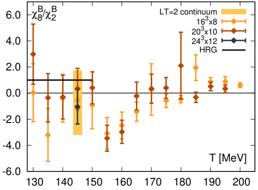


Chiral observables suffer from the reduced volume, but heavy quark observables (e.g. Polyakov loop) are much less affected. *[Wuppertal-Budapest 2405.1232]*

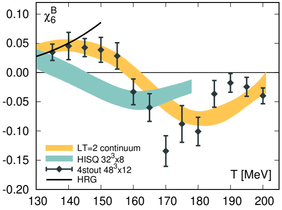
4Hex continuum result [Phys.Rev.D 110 (2024) 1, L011501]

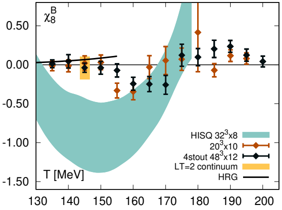






Comparison with literature





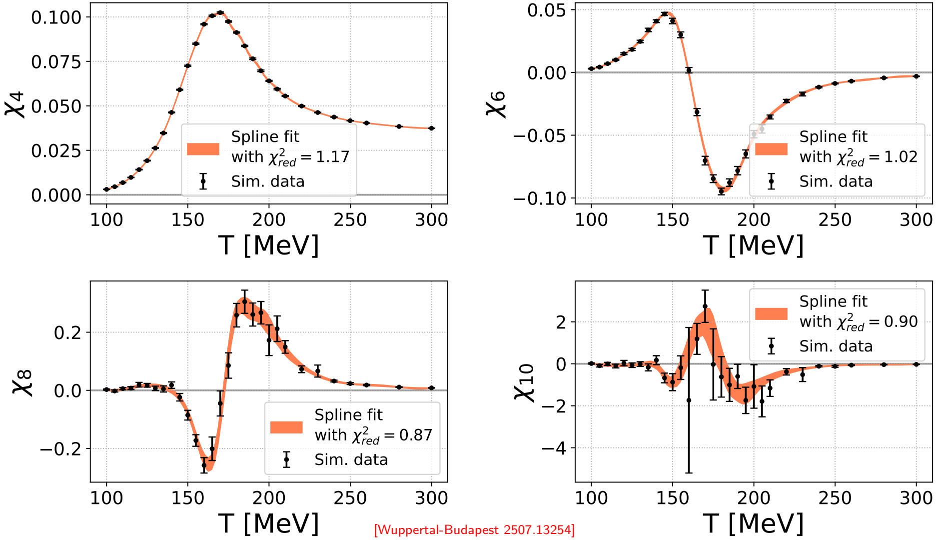
4stout data (imaginary μ_B, $48^3 \times 12$ lattice) : [Wuppertal-Budapest, 1805.04445]

HISQ data: ($\mu_B = 0$, $32^3 \times 8$ lattice, inexact charge conservation) [BNL-Bielefeld, 2202.09184,2212.09043]

$16^3 \times 8$ lattice, 4HEX action, full μ-dependence is recorded.

T [MeV]	# configurations at $\mu_B = i\pi(j/8)$								
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$
100	0.9M	-	-	-	-	-	-	-	-
105	0.9M	-	-	-	-	-	-	-	-
110	0.8M	-	-	-	-	-	-	-	-
115	1.1M	-	-	-	-	-	-	-	-
120	1.1M	-	-	-	-	-	-	-	-
125	1.7M	-	-	-	-	-	-	-	-
130	2.1M	-	96k	-	96k	-	96k	-	96k
135	1.5M	111k	105k	104k	120k	103k	104k	103k	103k
140	4.0M	100k	105k	105k	359k	104k	99k	103k	114k
145	2.6M	107k	106k	105k	122k	104k	103k	102k	115k
150	1.7M	114k	114k	112k	117k	110k	98k	96k	109k
155	1.6M	109k	109k	107k	126k	104k	103k	101k	116k
160	2.2M	112k	111k	109k	128k	104k	96k	101k	109k
165	1.8M	104k	102k	90k	121k	96k	93k	87k	105k
170	1.2M				-				

Baryon cumulants from $16^3 \times 8$ simulations

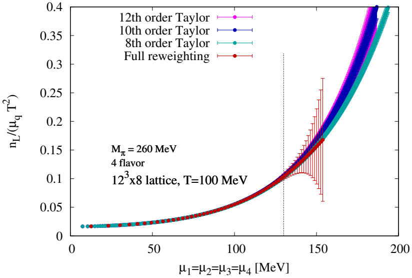


[Wuppertal-Budapest 2507.13254]

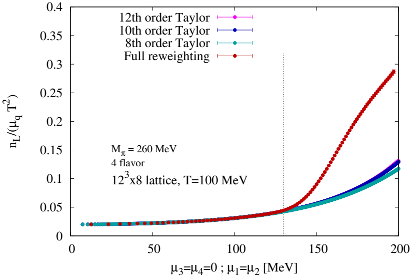
Reweighting to $\mu_B > 0$ fails for staggered quarks

Let us attempt a reweighting from $\mu_B = 0$ in the low temperature phase.

4 flavors, no rooting at $\mu_B > 0$

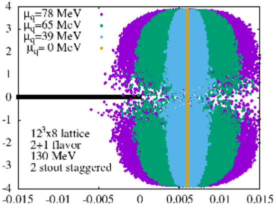


2+2 flavors, rooting at $\mu_B > 0$



The staggered eigenvalues come in quartets, near the continuum, their distance is $\mathcal{O}(a)$. The cut of the square root separates the eigenvalues that belong together, randomizing the phase of the determinant.

[Wuppertal-Budapest [2308.06105]]



Canoncial simulations of QCD could give us access to finite density without extrapolations.

- A. Hasenfratz, D. Toussaint (1992)
  Reduced matrix, $\mu_B = 0 \longrightarrow \text{canoncial}$ reweighting.
- A. Alexandru, M. Faber, I. Horváth, K-F Liu
  Sign quenched simulation + reweighting
- S. Kratochvila, P. de Forcrand (2006)
  Reduced matrix, $\mu_B = 0 \longrightarrow \text{canoncial}$ reweighting.
- S. Kratochvila, P. de Forcrand (2006)
  Canonical simulation at $B = 0$ with Z_3 moves for center symmetry
- S. Ejiri (2008)
  DOS+Reweighting to canonical with approximations to beat the sign problem
- J. Danzer, C. Gattringer (2012)
  $\mu_B = 0 \longrightarrow \text{canoncial}$ reweighting, than back to GCE
- A. Li, A. Alexandru, K.F Liu, X Meng (2015)
  Sign quenched simulation + reweighting, Winding number expansion
- A. Nakamura, S. Oka & Y. Taniguchi (2015)
  Ferrenberg-Swendsen reweighting, multiplecision integration

Let us consider here the reweighting approach from $\mu = 0$:

1. Simulate a huge statistics at $\mu_B = 0$.
   (Our case: $16^3 \times 8$ lattice, several million configurations)
2. Compute the quark determinants at imaginary quark determinants *(all real, rooting is fine)*

$$D_q(\varphi_q) \quad = \quad \left(\frac{\det M_{KS}(U, m_{ud}, \varphi_q)}{\det M_{KS}(U, m_{ud}, 0)}\right)^{1/2} \left(\frac{\det M_{KS}(U, m_s, \varphi_q)}{\det M_{KS}(U, m_s, 0)}\right)^{1/4}$$

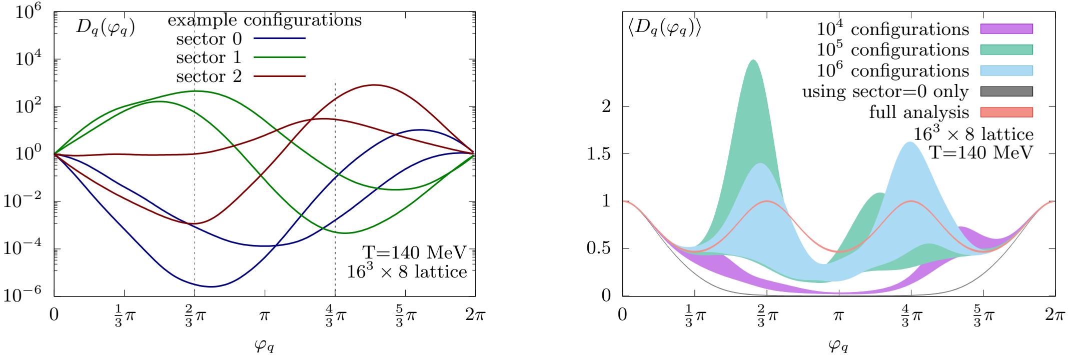
3. Perform the reweighting to obtain $Z_{GC}(T, V, \mu = i\varphi T)$:

$$\frac{Z_{GC}(T, V, \mu_q = i\varphi_q T)}{Z_{GC}(T, V, \mu_q = 0)} = \langle D_q(\varphi_q)\rangle$$

4. Find $Z_C(T, V, N)$ by Fourier transforming $Z_{GC}(T, V, \mu = i\varphi T)$.

$$Z_C(T, V, N) = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i\varphi N} Z_{GC}(T, V, i\varphi T)$$

- $D_q(\varphi_q)$ is real, but has has no symmetry other than the trivial periodicity in 2π.
- $\langle D_q(\varphi_q)\rangle$ has a period of $2\pi/N_c$ *a manifestation of center symmetry*.
- Do data show this symmetry?



The observable $\langle D_q(\varphi_q)\rangle_{\mu=0}$ suffers from an overlap problem.

Statement: The red curve in the right panel (full result) is
= the black curve + black curve shifted forward + back curve shifted backward

Idea: we restrict simulations to a single sector

The three sectors are mathematically equivalent.

The canonical ensemble does not break center symmetry (unlike $\mu_B = 0$).

We rewrote

$$\langle D_q(\varphi_q)\rangle = \frac{Z_{S0}\langle D_q(\varphi_q)\rangle_{S0} + Z_{S1}\langle D_q(\varphi_q)\rangle_{S1} + Z_{S2}\langle D_q(\varphi_q)\rangle_{S2}}{Z_{S0}+Z_{S1}+Z_{S2}}$$

into

$$\langle D_q(\varphi_q)\rangle = \frac{Z_{S0}\langle D_q(\varphi_q)\rangle_{S0} + Z_{S0}\langle D_q(\varphi_q+2\pi/3)\rangle_{S0} + Z_{S0}\langle D_q(\varphi_q-2\pi/3)\rangle_{S0}}{Z_{S0}+Z_{S1}+Z_{S2}}$$

The normalization can be easily figured out:

$$\frac{Z_{S1,S2}}{Z_{S0}} = \langle D_q(\pm 2\pi/3)\rangle_{S0}$$

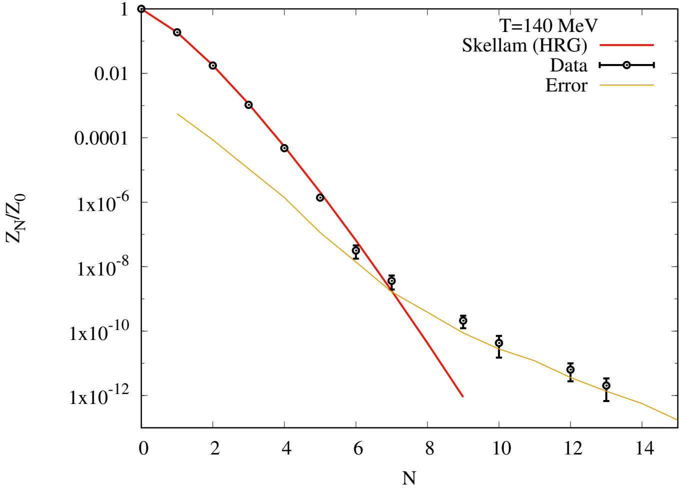
Sector definition: where is the maximum of $D_q(\varphi_q)$.

On our $16^3 \times 8$ lattices, the weight of sector zero was

- $T = 130$ MeV: 96.9 %
- $T = 160$ MeV: 99.9 %

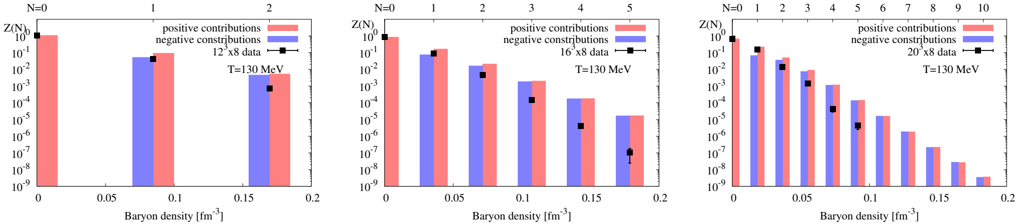
Using only sector-zero observables dramatically reduces the error.

The canonical partition sum mostly follows the Skellam distribution at low temperature.



The imagiry part cancels between U and U^*.
The real part of the canonical determinants is highly oscillating.

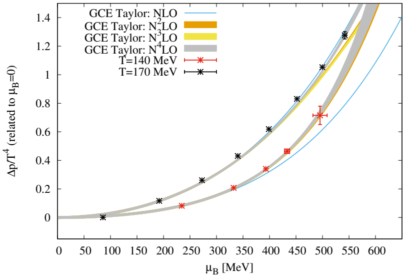
Here we show the positive and negative contributions and the value after cancellation.

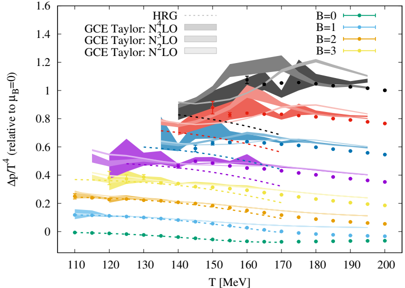


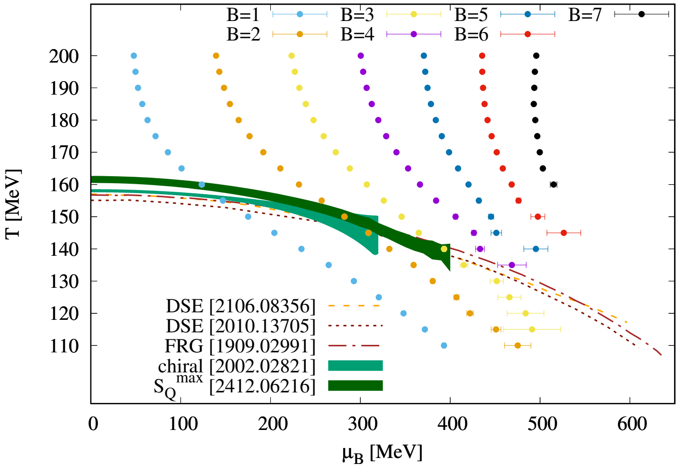
Observe how the severity increases with volume.

Observables:

$$
\begin{aligned}
F(T,V,N) &= -T\log Z_C(T,V,N)\\
\mu_B &= F(T,V,N) - F(T,V,N-1)\\
p_C(T,V,N) &= -\frac{\partial F(T,V,N)}{\partial V} = -\frac{1}{V}\left.\frac{dF(T,\alpha V,N)}{d\alpha}\right|_{\alpha=1}\\
&= -\frac{T}{V}\frac{\int_0^{2\pi}\frac{d\varphi}{2\pi}\Omega(i\varphi T)e^{-i\varphi N - T^{-1}\Omega(T,V,i\varphi T)}}{\int_0^{2\pi}\frac{d\varphi}{2\pi}e^{-i\varphi N - T^{-1}\Omega(T,V,i\varphi T)}}\ .
\end{aligned}
$$






B=1
B=2
B=3
B=4
B=5
B=6
B=7
T [MeV]
200
190
180
170
160
150
140
130
120
110
0
100
200
300
400
500
600
μ_B [MeV]
DSE [2106.08356]
DSE [2010.13705]
FRG [1909.02991]
chiral [2002.02821]
S_Q^max [2412.06216]

- Frequently used, but not useful *(difficult to justify the truncation)*

$$e^{-\Omega/T} = Z_{GC}(T,V,\mu) = \sum_{N=-\infty}^{\infty} e^{\frac{\mu}{T}N} Z_C(T,V,N)$$

- Helmholz free energy through Legendre transfrom:

$$F_{LT}(T,V,\langle N\rangle) = \Omega(T,V,\mu) + \mu\langle N\rangle\,, \qquad \text{with} \quad \langle \mathrm{N}\rangle = -\frac{\partial\Omega}{\partial\mu}$$

  Here $\langle N\rangle$ does not need to be an integer.
- $F_{LT} = \Omega + \mu\langle N\rangle$ and $F = -T\log Z_C$ do not agree in a finite volume.

Can we compute F_{LT} from F on a finite volume?

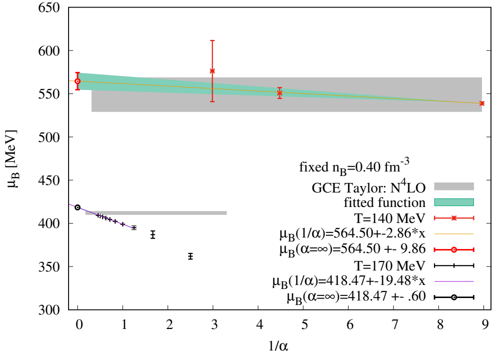
$$\frac{1}{\alpha}F(T,\alpha V,\alpha N) \;=\; F_{LT}(T,V,N) + \frac{T}{2\alpha}\log\left(2\pi\alpha\chi_2(\mu^\star)VT^3\right)\left(1+\mathcal{O}(\alpha^{-1})\right)\,,$$

Strategy:

- Compute $\frac{1}{\alpha}F(T,\alpha V,\alpha N)$ for several α
- Make the $\alpha\to\infty$ limit

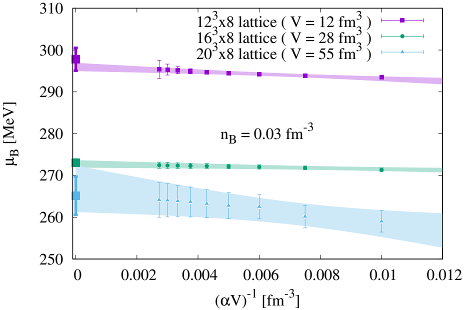
This is an extrapolation in $\alpha \to \infty$ at constant density!

Extrapolation at two different temperatures:

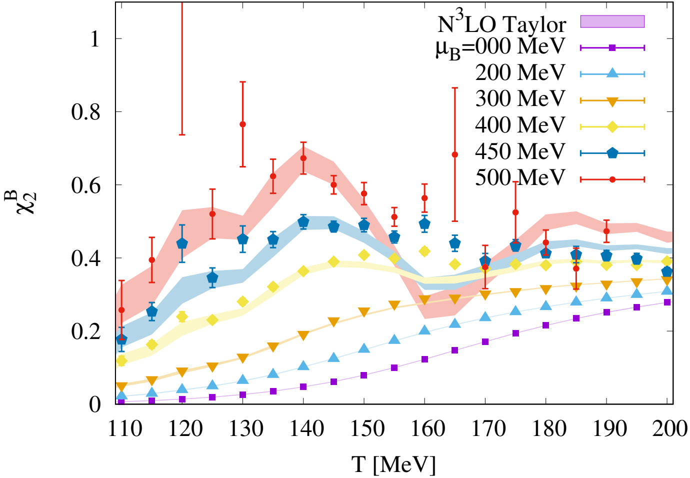


The ideal range of the extrapolation is temperature dependent.
For low T we often only use $N = 1, 2, 3$ where $\alpha < 1$.

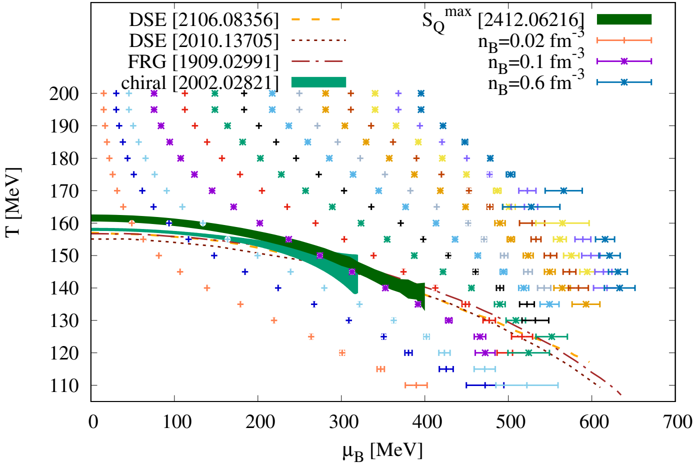
Three volumes:



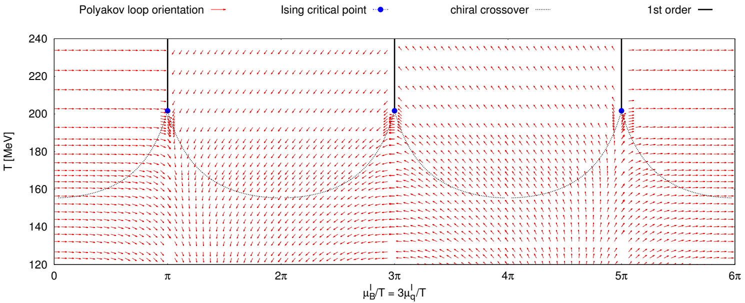
The extrapolated grand canonical result gives back the grand canonical (Taylor expanded) value of μ_B. The $\alpha \to \infty$ limit is not an infinite volume limit, it reproduces the finite volume effects of the original grand canonical ensemble.



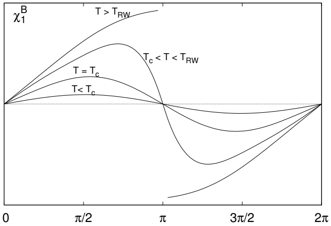
[Wuppertal-Budapest 2604.14117]



[Wuppertal-Budapest 2604.14117]



$T_{\rm RW} = 208(5)$ MeV [Bonati et al, 1602.01426]



At low T only B states contribute:

$$\chi_1^B \sim \sin(\mu_B/T)$$

Then $\chi_4^B/\chi_2^B = 1$ as we find in simulations.

At high T fractional charges are required to make a first order transition at $\mu_B = \pi T$:

$$\chi_1^B \sim \sin(\mu_B/T/3)$$

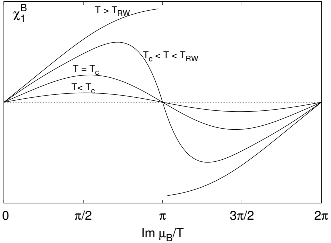
Exact symmetries of the QCD thermodynamic potential $\log Z \approx pV$:

$$
\begin{aligned}
p(T,\mu_B) &= p(T,\mu_B + i2\pi T) && \textit{Center symmetry} \\
p(T,\mu_B) &= p(T,-\mu_B) && \textit{Charge conjugation symmetry}
\end{aligned}
$$

With $\hat{p} = p/T^4$ and $\hat{\mu}_B = \mu_B/T$ we can define the fugacity series

$$\hat{p}(T,\hat{\mu}_B) = p_0 + p_1 \cosh(\hat{\mu}_B) + p_2 \cosh(2\hat{\mu}_B) + p_3 \cosh(3\hat{\mu}_B) + \dots$$

which is the Fourier series of the pressure in imaginary μ_B.



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At high T fractional charges are required to make a first order transition at $\mu_B = \pi T$:

$$\chi_1^B \sim \sin(\mu_B/T/3)$$

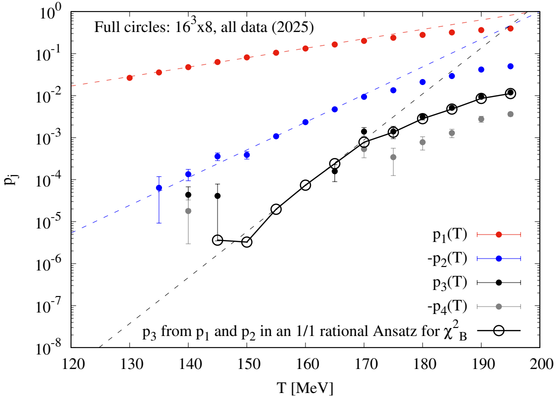
Is this an alternative to Taylor?

No straightforward truncation.

(too) simple rational model:

$$\chi_2^B(\hat{\mu}_B) \sim \frac{\cosh(\hat{\mu}_B)}{1+\epsilon(T)\cosh(\hat{\mu}_B)}, \qquad \text{implies} \quad \frac{p_3}{p_2} \approx 1.78\frac{p_2}{p_1}, \quad \frac{p_2}{p_1} = -\frac{\epsilon(T)}{4}$$

Other models (e.g. cluster expansion [Vovchenko 1711.01261]) differ in the temperature independent coefficient.



$p_1(T)$: baryon abundance,

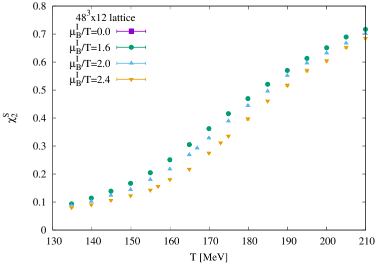
$p_2(T)/p_1(T)$: baryon interactions,

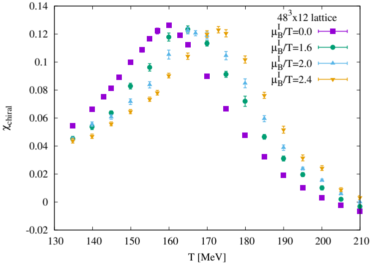
$p_3(T)/p_1(T)$: two-body + three-body effects

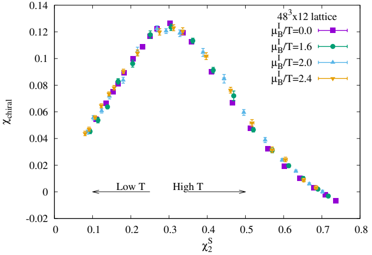
Goal: calculate $p_3(T)$ - model prediction.

An observation about T_c and strangeness fluctuations

- Chiral susceptibility peaks at T_c
- The peak shifts with μ_B as expected
- Strange susceptibility $\chi_2^S(T)$ is monotonic in T
- *Collapse plot:* chiral susceptibility as a function of starngeness susceptibility is μ_B independent







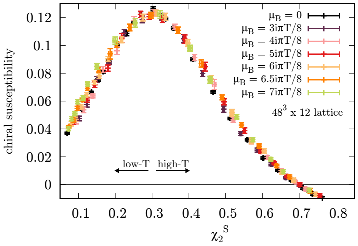
Lession from imaginary μ_B: $\chi_2^S \approx 0.3$ marks T_c for various μ_B

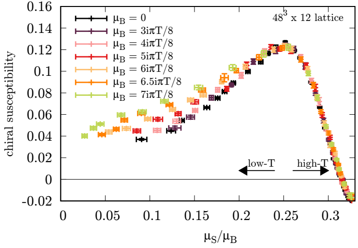
The behaviour is specific to the strangeness neutral case.

Two proposed proxies for the $T_c(\mu_B)$ line:

$$\chi_S(T_c,\mu_B) \approx 0.3$$
$$\frac{d\mu_S}{d\mu_B} \approx 0.25$$

based on the observation at **zero and imaginary** μ_B.

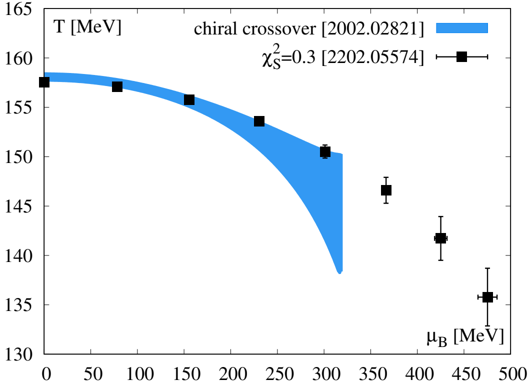




[Wuppertal-Budapest 2510.26455]

Constant $\chi_2^S(T, \mu_B)$ contour vs the crossover line

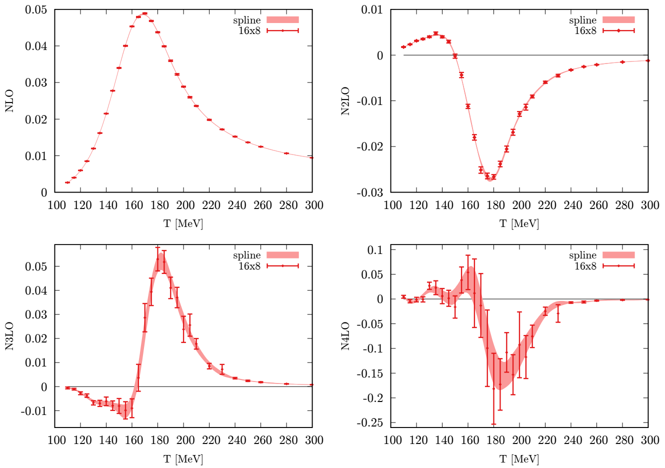
We can check the agreement between the $\chi_2^S = 0.3$ contour and the extrapolated chiral transition line with **published continuum data**.



There is a small tension between the curvatures, but line is within 1 σ.
$\chi_2^S(T)$ is a result of NLO T' expansion
T_c (chiral) is the cubic polynomial extrapolation of the peak of $\chi_{\bar{\psi}\psi}$.

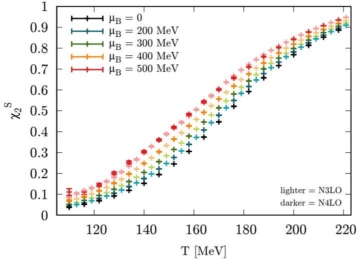
Taylor expansion coefficients with strangeness neutrality

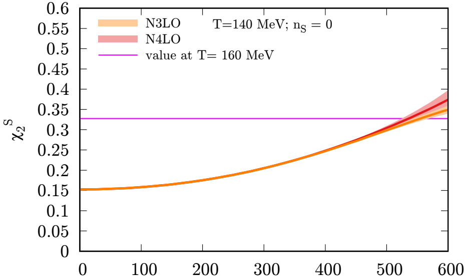
$$\chi_2^S(\mu_B, \mu_S^*(\mu_B)) = \chi_2^S + \underbrace{\left(\chi_{22}^{BS} + 2\chi_{13}^{BS} s_1(T) + \chi_{04}^{BS}(s_1(T))^2\right)}_{\chi^S_{2,NLO}} \hat{\mu}_B^2 + \dots$$

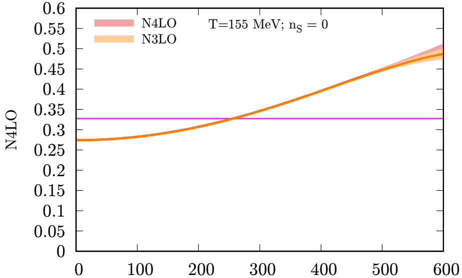


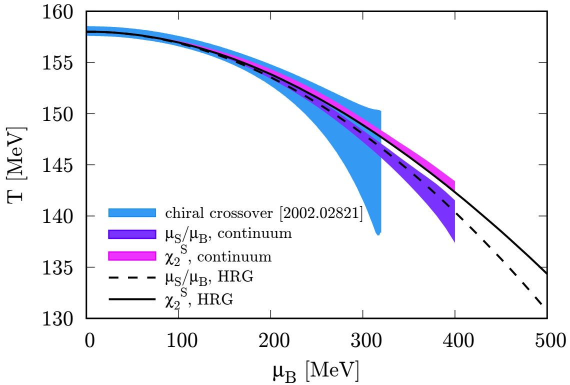
$16^3 \times 8$ lattice, 4HEX, (Wuppertal-Budapest)

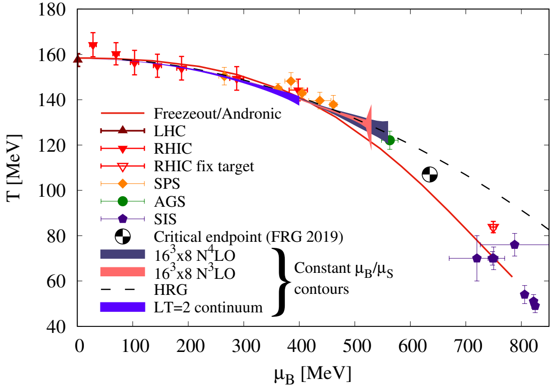
Taylor extrapolation from $\mu_B = 0$ of χ_2^S:





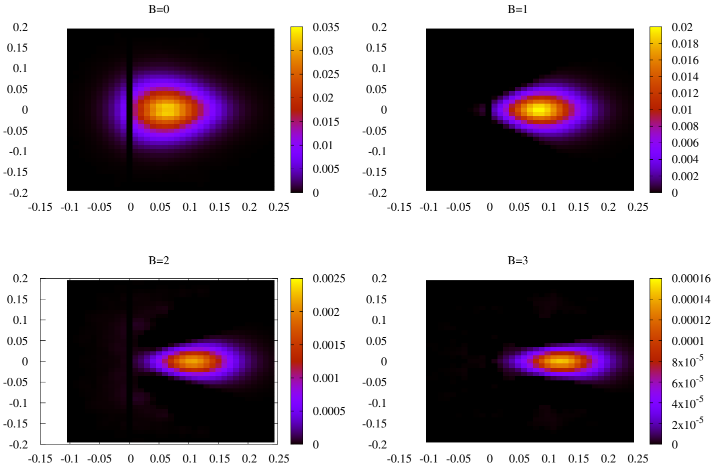




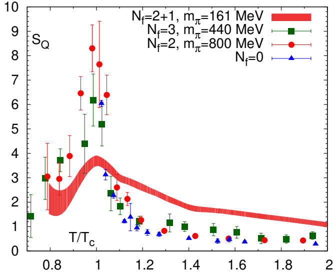


[Wuppertal-Budapest 2510.26455]

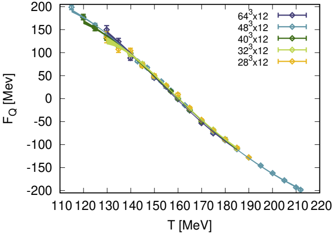
$T = 140$ MeV, $16^3 \times 8$, canonical ensemble in baryon number.

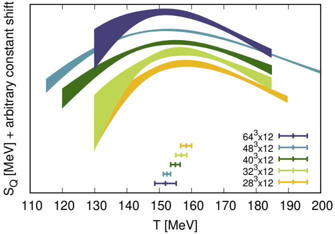


$$P = \frac{1}{V}\left\langle \mathrm{Tr}\prod_i U_i \right\rangle$$
$$F_Q = -T\log P$$
$$S_Q = -\frac{\partial F_Q(T,\mu_B)}{\partial T}$$
T_c : *peak of S_Q in T*



[Left: TUM QCD, Bazavov et al 1603.06637]



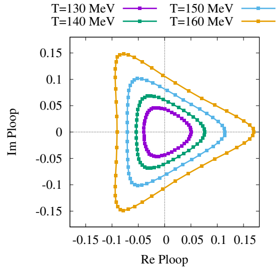


[Right: Wuppertal-Budapest 2405.12320]

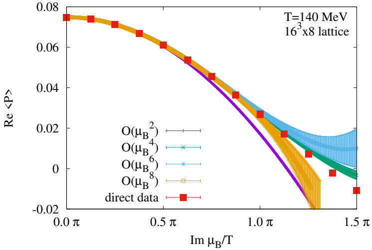
$$P = P_E + P_O, \quad \bar{P} = P_E - P_O, \quad P\bar{P} = P_E^2 - P_O^2 = |P|^2$$

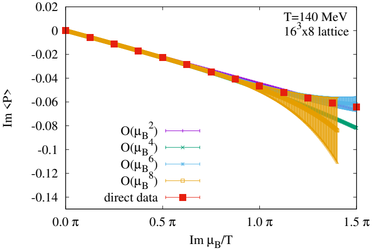
Magenta color: C-even (real).: $P_E = \mathrm{Re}\, P$

Blue color: C-odd (imaginary): $P_O = i\, \mathrm{Im}\, P$



$$\begin{aligned}
\partial_j \langle P_E \rangle|_{\mu\equiv 0} &= 0\,, \\
\partial_j \langle P_O \rangle|_{\mu\equiv 0} &= \langle A_j P_O \rangle\,, \\
\partial_j \partial_k \langle P_E \rangle|_{\mu\equiv 0} &= \delta_{jk}\left[\langle B_j P_E \rangle - \langle B_j \rangle \langle P_E \rangle\right] \\
&\quad \langle A_j A_k P_E \rangle - \langle A_j A_k \rangle \langle P_E \rangle\,, \\
\partial_j \partial_k \langle P_O \rangle|_{\mu\equiv 0} &= 0\,.
\end{aligned}$$





$$P = P_E + P_O, \quad \bar{P} = P_E - P_O, \qquad P\bar{P} = P_E^2 - P_O^2$$

$\langle P \rangle^* \neq \langle \bar{P} \rangle$ the probabilities are complex! $(\langle P \rangle \cdot \langle \bar{P} \rangle \neq |\langle \bar{P} \rangle|^2)$)

P_E is even and real

P_O is odd, imaginary but $\langle P_O \rangle$ is **real** and odd

$$\begin{aligned}
\partial_j \langle P_E \rangle\big|_{\mu\equiv 0} &= 0\,, \\
\partial_j \langle P_O \rangle\big|_{\mu\equiv 0} &= \langle A_j P_O \rangle\,, \\
\partial_j \partial_k \langle P_E \rangle\big|_{\mu\equiv 0} &= \delta_{jk} \left[\langle B_j P_E \rangle - \langle B_j \rangle \langle P_E \rangle \right] \\
& \quad \langle A_j A_k P_E \rangle - \langle A_j A_k \rangle \langle P_E \rangle\,, \\
\partial_j \partial_k \langle P_O \rangle\big|_{\mu\equiv 0} &= 0\,.
\end{aligned}$$

If ∂_j is a **real chemical potential** derivative:

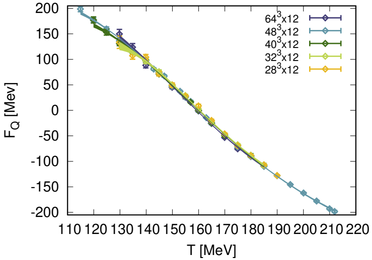
$$\langle P_O \rangle_{\mu_B} = \underbrace{\langle A_j P_O \rangle_{\mu_B = 0}}_{\text{real}} \mu_B + \dots\,,$$

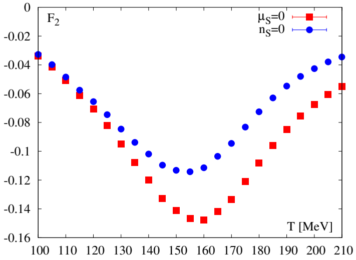
$\langle P_O \rangle$ was meant to be the imaginary part, but in fact,
it has a ***real expectation value****, while being C-odd.*

We can extrapolate either $\langle P_E \rangle$ or $\langle P \rangle \cdot \langle \bar{P} \rangle$, at $\mu_B = 0$ they were equivalent.

$$F = -\frac{T}{2} \log(\langle P \rangle \langle \bar{P} \rangle) = \frac{F_Q + F_{\bar{Q}}}{2} = T \sum_{n=0,2,\ldots} \frac{F_n}{n!} \left(\frac{\mu_B}{T}\right)^n$$

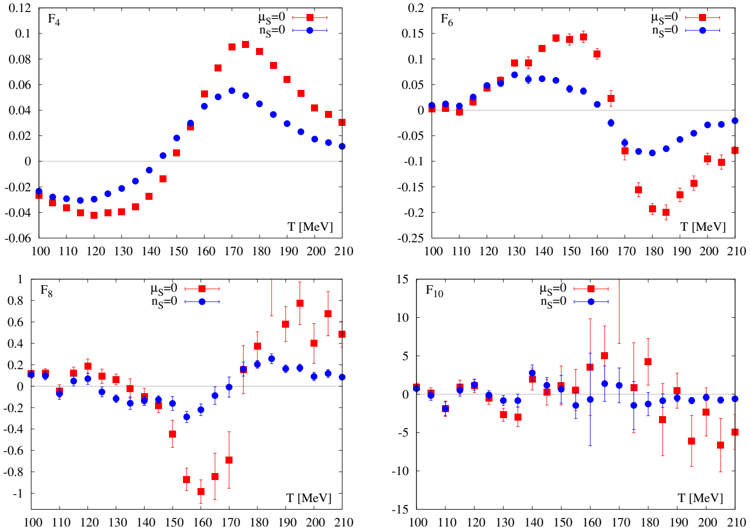
$$F_0 = -\frac{1}{2} \log(\langle P \rangle \langle \bar{P} \rangle) \qquad F_2 = -\frac{1}{2 \langle P \rangle \langle \bar{P} \rangle} \frac{\partial^2}{(\partial \mu_B / T)^2} \langle P \rangle \langle \bar{P} \rangle$$



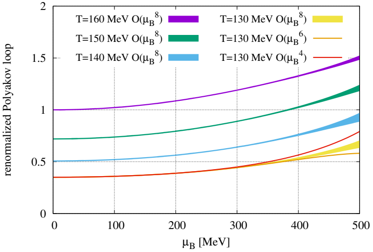


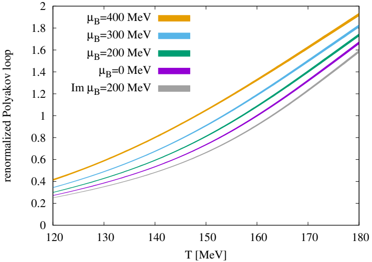
[This result: Wuppertal-Budapest 2410.06216, first computed: D'Elia et al 1907.09461]

Higher orders:



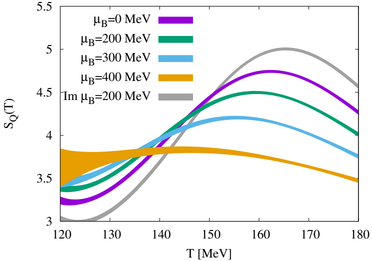
The renoramalized Polyakov loop:





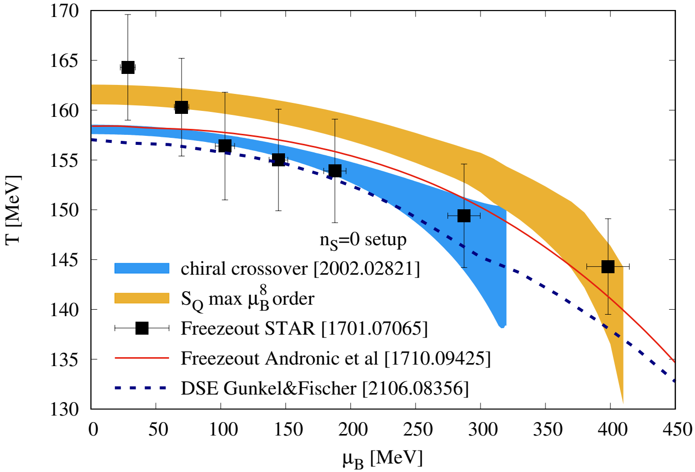
$$F_Q = -\frac{T}{2}\log(|P|^2)$$

$$S_Q = -\frac{\partial F_Q(T)}{\partial T}$$

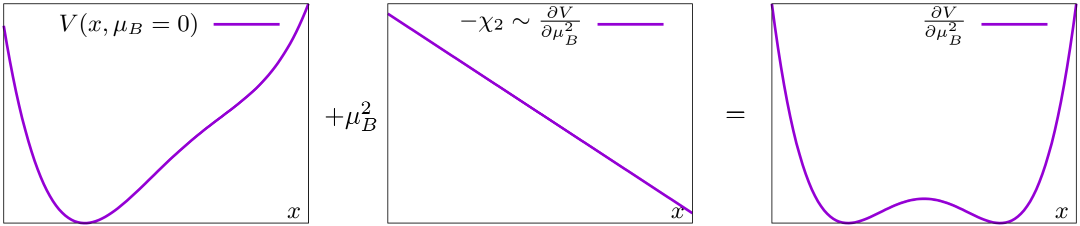


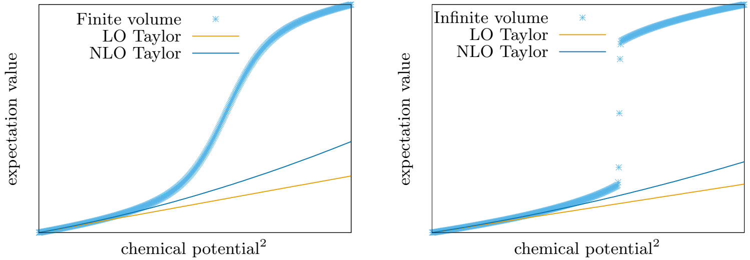
[Wuppertal-Budapest 2410.06216]

S_Q itself can be extrapolated *($16^3 \times 8$ lattices)*.
The temperature of the S_Q-peak is calculated for each μ_B.

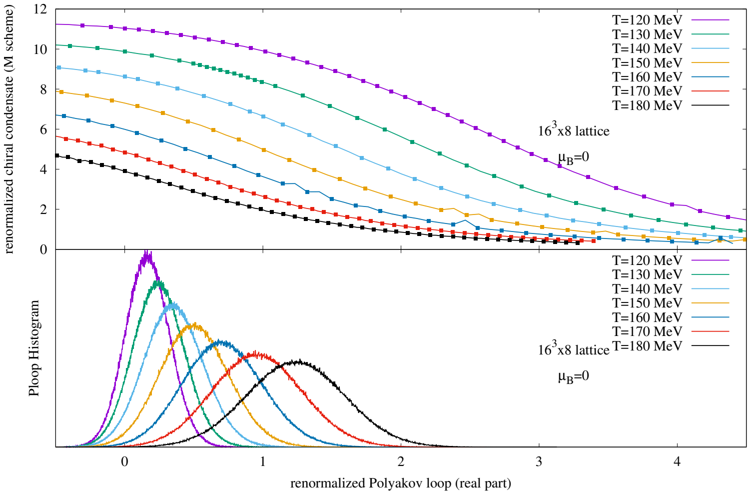


Effective potential vs. expectation values





x: your favourite order parameter



- Polyakov loop distribution shifts and broadens with temperature
- $\langle\bar{\psi}\psi\rangle$ and Polyakov loop are linked
- $\langle\bar{\psi}\psi\rangle$ depends on temperature by itself and also through the Polyakov loop

1. We expand $V_{\text{eff}}(x)$ at fixed x in μ_B^2.

$$-V_{\text{eff}}(x,\mu_B) = \chi_0(x) + \frac{\hat{\mu}_B^2}{2!}\chi_2(x) + \frac{\hat{\mu}_B^4}{4!}\chi_4(x) + \frac{\hat{\mu}_B^6}{6!}\chi_6(x) + \frac{\hat{\mu}_B^8}{8!}\chi_8(x) + \dots$$

We do not extrapolate observables, but the effective potential.

2. Let's simulate $\chi_n(x)$ at fixed x with constrained simulations.

$$\chi_n(x) = \langle \chi_n \rangle_x$$

We include high-x configurations, that are preferred at high μ_B, but are not sampled at $\mu_B = 0$.

Density of States formalism.

3. Observables are computed from

$$Z(\mu_B) = \int_{-\infty}^{\infty} dx e^{-(LT)^3 V_{\text{eff}}(x,\mu_B)}$$

This $Z(\mu_B)$ can be zero. $\rightarrow$ Lee-Yang zeros can be computed.

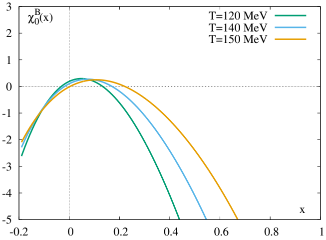
$$\chi_{2n} = \frac{\partial^{2n} \log Z(\mu_B)}{\partial \hat{\mu}_B^{2n}} = \frac{\partial^{2n}}{\partial \hat{\mu}_B^{2n}} \log \int_{-\infty}^{\infty} dx e^{-(LT)^3 V_{\text{eff}}(x,\mu_B)} \quad \longrightarrow$$ *Equation of state & fluctuations*

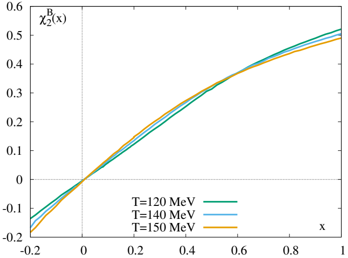
Constrained expansion coefficiens

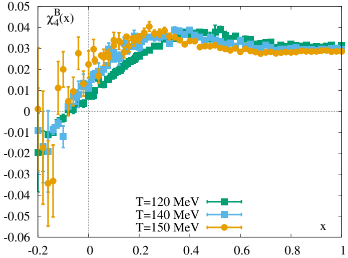
For a fixed temperature T

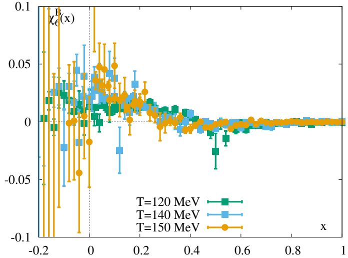
$$-V_{\text{eff}}(x, \mu_B) = \chi_0 + \frac{\hat{\mu}_B^2}{2!}\chi_2(x) + \frac{\hat{\mu}_B^4}{4!}\chi_4(x) + \frac{\hat{\mu}_B^6}{6!}\chi_6(x) + \ldots$$

Fixed-x coefficients drop faster than in ordinary Taylor expansion.

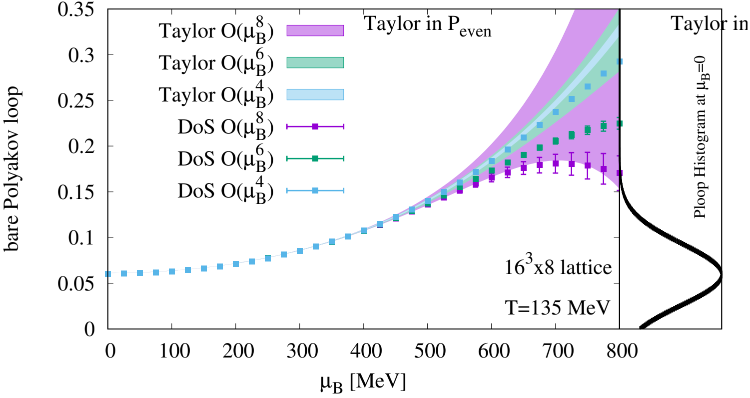


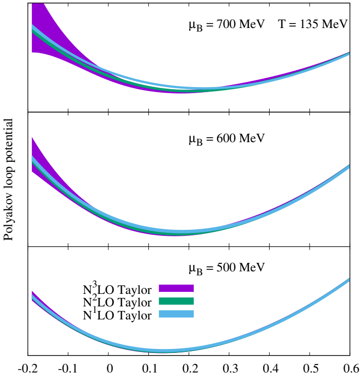






- We extrapolate the Polyakov loop's expectation value.
  Taylor method [Wuppertal-Budapest 2410.06216]
- We caluclate the μ_B-dependent potential.
  density of states method





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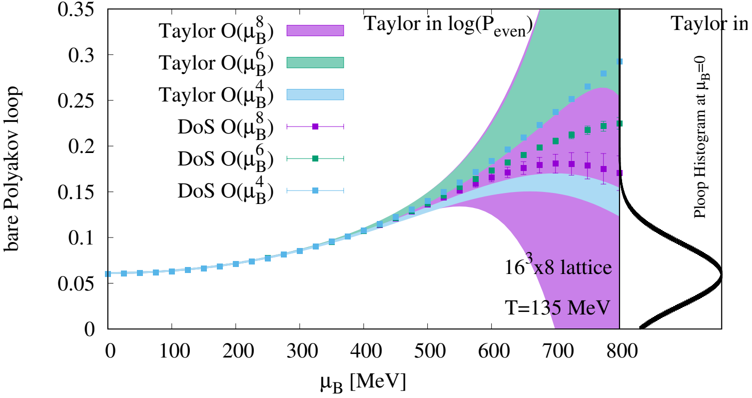
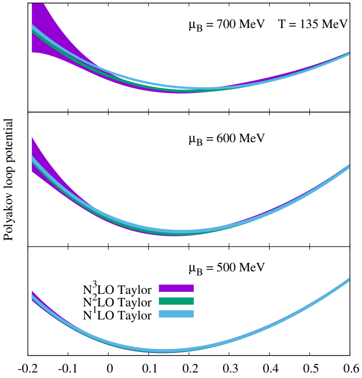
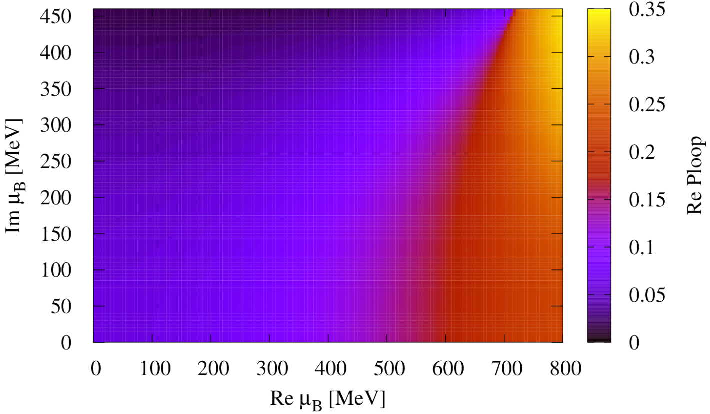




The complex μ_B plane at $T = 135$ MeV

We insert a complex μ_B into the formula

$$-V(x,\mu_B) = \chi_0 + \frac{\hat{\mu}_B^2}{2!}\chi_2(x) + \frac{\hat{\mu}_B^4}{4!}\chi_4(x) + \frac{\hat{\mu}_B^6}{6!}\chi_6(x) + \dots$$



The minimum of the potential $V(x,\mu_B)$ is color coded. (x=Re Ploop)

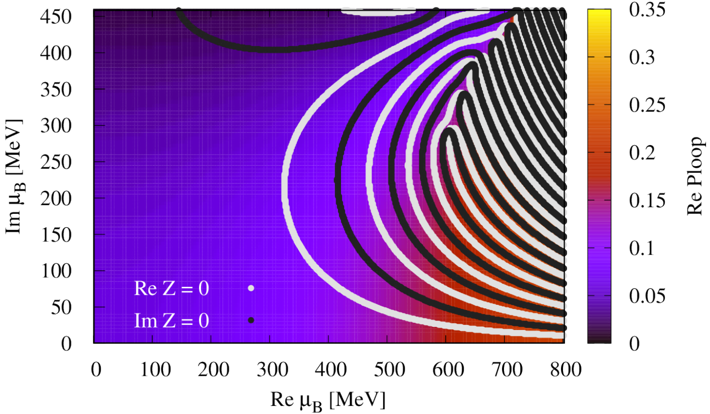
$$Z(\mu_B) = \int_{-\infty}^{+\infty} dx e^{-(LT)^3 V(x,\mu_B)}$$

Lee-Yang zero:

$$Z(\mu_B) = 0$$

We insert a complex μ_B into the formula

$$Z(\mu) = \int_{-\infty}^{\infty} dx \exp\left((LT)^3 \left[\chi_0 + \frac{\hat{\mu}_B^2}{2!}\chi_2(x) + \frac{\hat{\mu}_B^4}{4!}\chi_4(x) + \frac{\hat{\mu}_B^6}{6!}\chi_6(x) \right] \right)$$

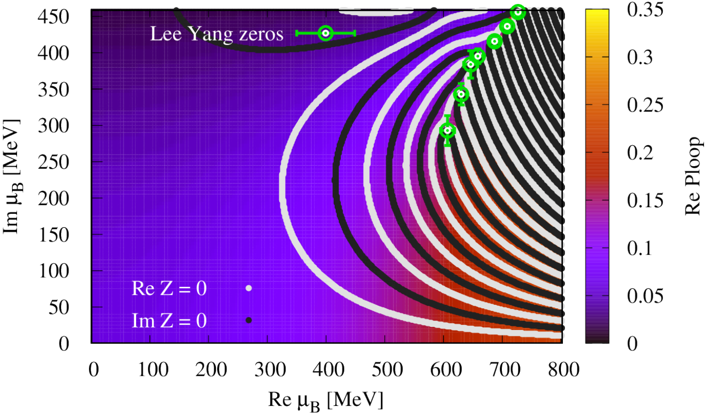


An algorithm is scanning the space for lines of $\mathrm{Re}\ Z = 0$ and lines of $\mathrm{Im}\ Z = 0$

These intersect in 90° angle at the Lee-Yang zeros.

We insert a complex μ_B into the formula

$$Z(\mu) = \int_{-\infty}^{\infty} dx \exp\left((LT)^3 \left[\chi_0 + \frac{\hat{\mu}_B^2}{2!}\chi_2(x) + \frac{\hat{\mu}_B^4}{4!}\chi_4(x) + \frac{\hat{\mu}_B^6}{6!}\chi_6(x) \right] \right)$$



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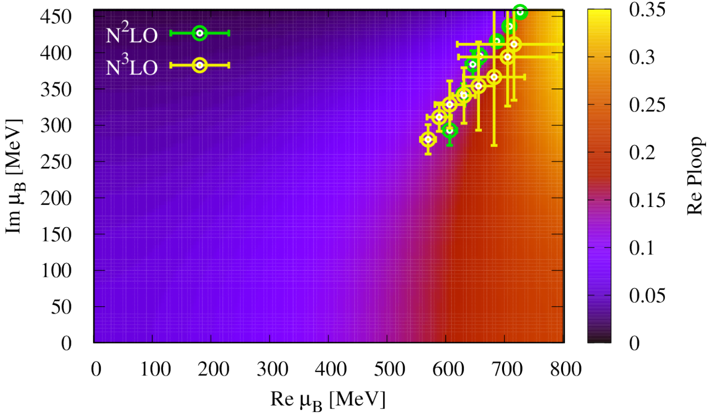
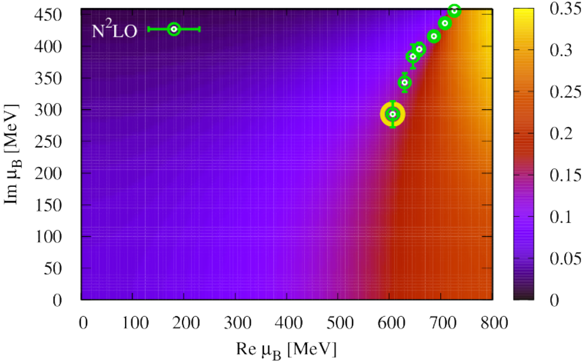
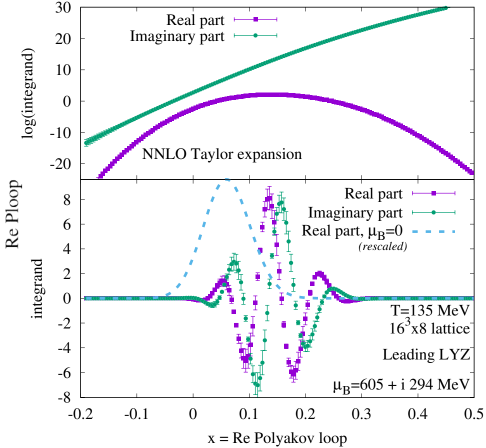


The terminal points of the Yang-Lee edges at N^{2}LO and N^3LO differ sligtly.

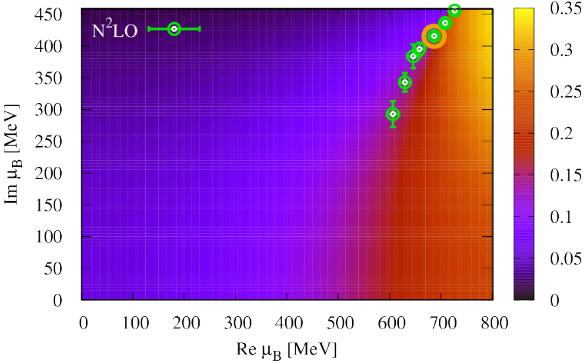
The complex μ_B plane at $T = 135$ MeV

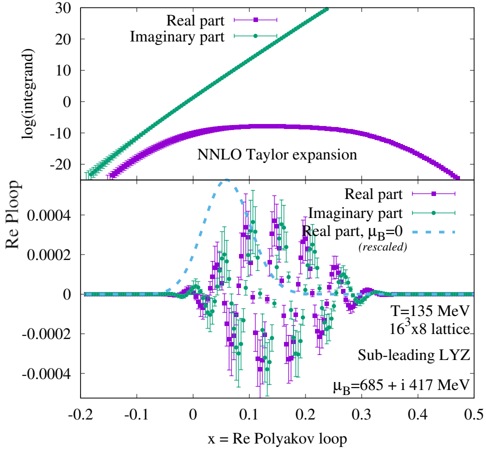
$$Z(\mu) = \int_{-\infty}^{\infty} dx \, \exp\left((LT)^3 \left[\chi_0 + \frac{\hat{\mu}_B^2}{2!}\chi_2(x) + \frac{\hat{\mu}_B^4}{4!}\chi_4(x) + \frac{\hat{\mu}_B^6}{6!}\chi_6(x) \right] \right)$$





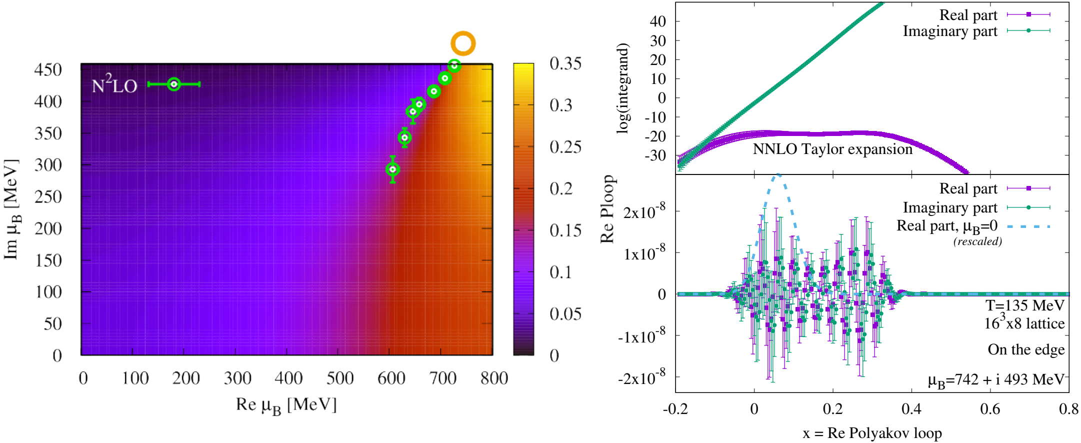
$$Z(\mu) = \int_{-\infty}^{\infty} dx \, \exp\left((LT)^3 \left[\chi_0 + \frac{\hat{\mu}_B^2}{2!}\chi_2(x) + \frac{\hat{\mu}_B^4}{4!}\chi_4(x) + \frac{\hat{\mu}_B^6}{6!}\chi_6(x) \right] \right)$$



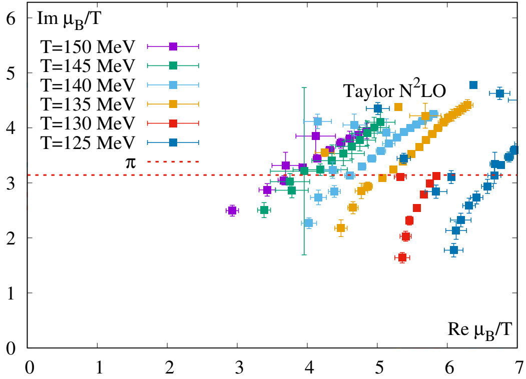


The complex μ_B plane at $T = 135$ MeV

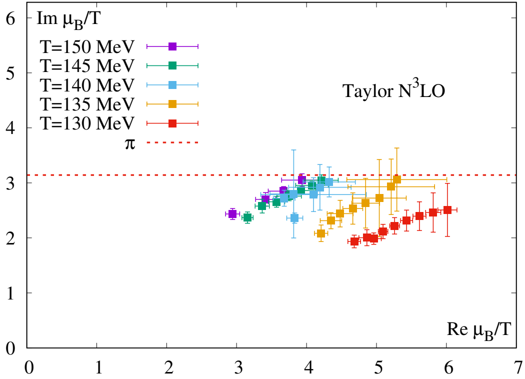
$$Z(\mu) = \int_{-\infty}^{\infty} dx \, \exp\left((LT)^3 \left[\chi_0 + \frac{\hat{\mu}_B^2}{2!}\chi_2(x) + \frac{\hat{\mu}_B^4}{4!}\chi_4(x) + \frac{\hat{\mu}_B^6}{6!}\chi_6(x)\right]\right)$$



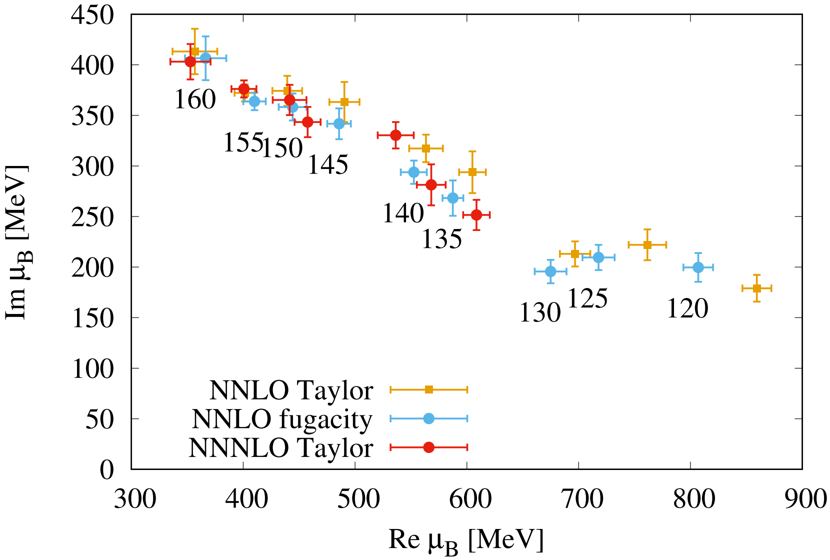
expansion up to $\mathcal{O}(\hat{\mu}_B^6)$

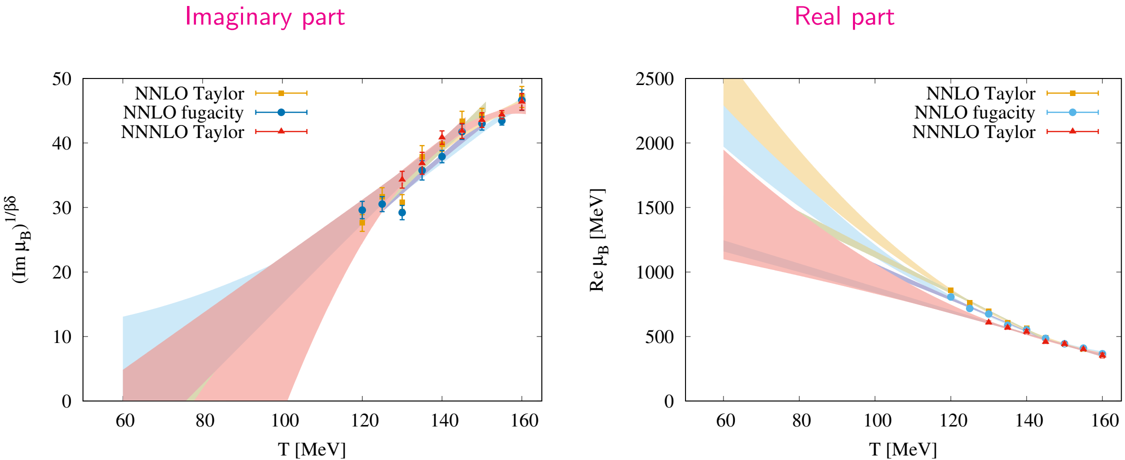


expansion up to $\mathcal{O}(\hat{\mu}_B^8)$



Lee-Yang zeroes in three expansion schemes

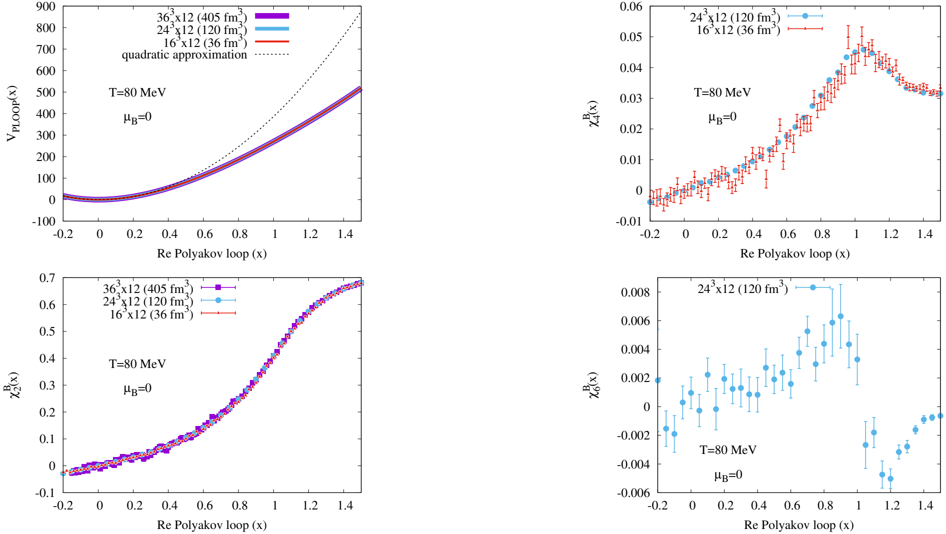




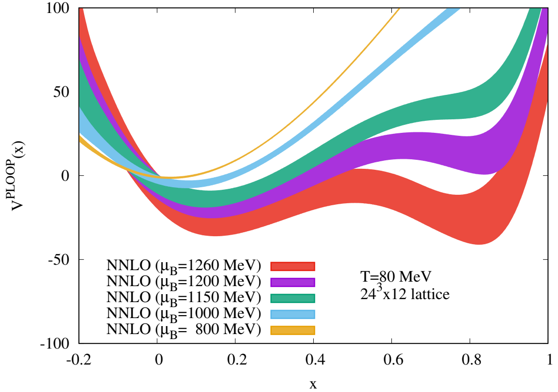
Extrapolation uncertainties are large as before.

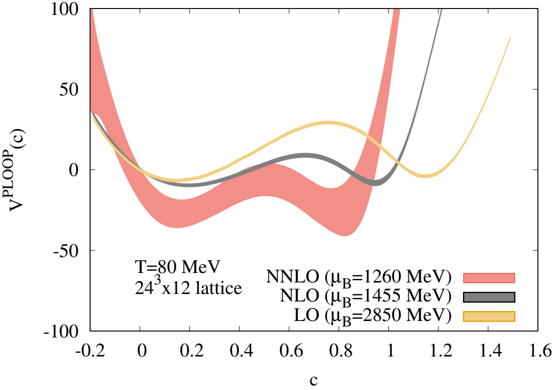
CEP is likely to be below $T < 100$ MeV and $\mu_B > 840$ MeV.

Directly at $T = 80$ MeV



2 million configurations on a $24^3 \times 12$ *lattice. No visible volume dependence up to NLO.*

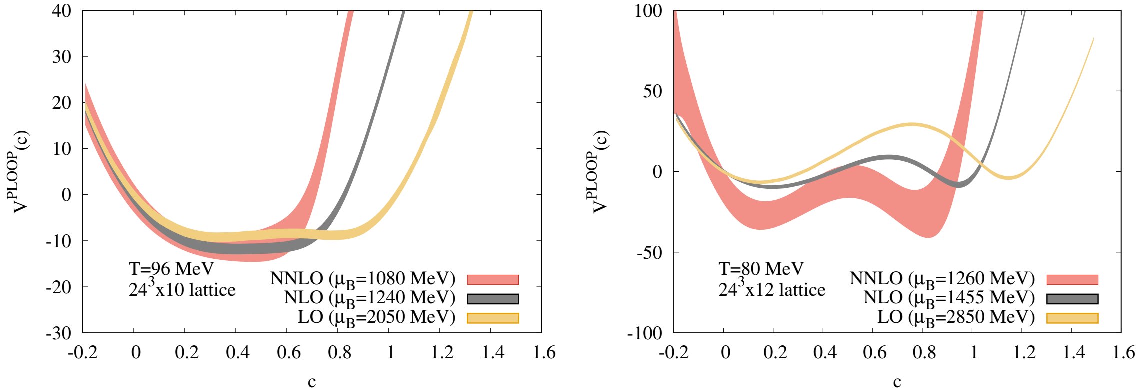




Three consecutive orders predict 1st order for this temperature.

(Not continuum extrapolated! $a = 0.2$ fm.)

Comparison $T = 96$ vs $T = 80$ MeV

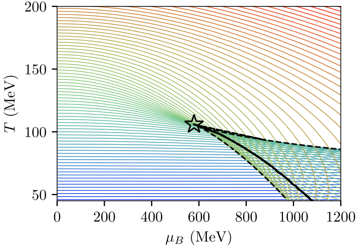


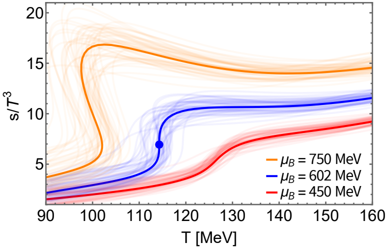
At 96 MeV the potential is very flat, $\mu_B(96\ \mathrm{MeV}) < \mu_B(80\ \mathrm{MeV})$

(Not continuum extrapolated! $a = 0.2$ fm.)

- We simulated the Polyakov-loop potential at zero and finite density
- We applied the density-of-states formalism and implemented constrained simulations
  right panel: baryon density through the 1st order transition
- Power series in μ_B^2 converges faster than Taylor expansion of observables.
- Lee-Yang zeros are accessible without rational modeling
- **If** higher orders and a continuum extarpolation do not change the conclusion:
  $80\ \mathrm{MeV} < T < 100\ \mathrm{MeV}$
  $840\ \mathrm{MeV} < \mu_B < 1250\ \mathrm{MeV}$
  (this is not meant to be a quantitative result and is preliminary)
- Higher orders and finer lattices are needed.

Idea: look at contours of constant entorpy/T^3, find spinodal regions.





[Black-Hole-Engineering model Hippert et al[2309.00579]] [QCD: Shah et al [2410.16206]]

Leading order: Entropy contours are exact parabolas: $T' = A + B\mu_B^2$

Optimistic assumption on error propagation.

Critical endpoint estimate: $T_c = 114.3 \pm 6.9$ MeV, $\mu_{B,c} = 602.1 \pm 62.1$ MeV

Is this a first principles result on a CEP?

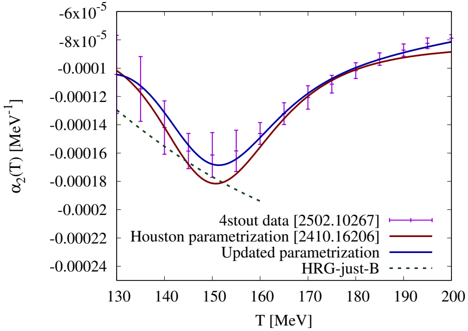
No, an expansion is defined, where

each order can be computed from first principles,

this expansion does not automatically break down near the CEP.

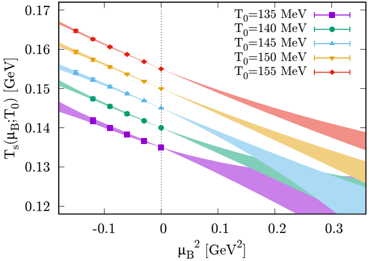
The Houston group introduced a sysematic scheme to expand the entropy [2410.16206]

$$s[T'(\mu_B;T),\mu_B] = s[T,\mu_B=0]$$
$$T'(\mu_B;T) = T+\alpha_2(T)\frac{1}{2}\mu_B^2+\dots$$
$$\mu_{B,c} = \sqrt{-2/\alpha_2'(T_*)}, \qquad \text{where} \quad \alpha_2''(T_*)=0$$



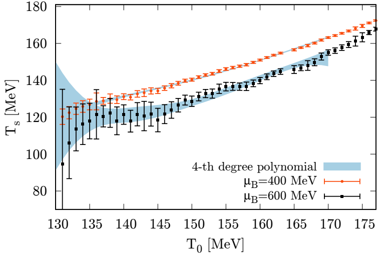
1. Calculate entropy at imaginary μ_B

2. Extrapolate constant s contours to $\mu_B > 0$

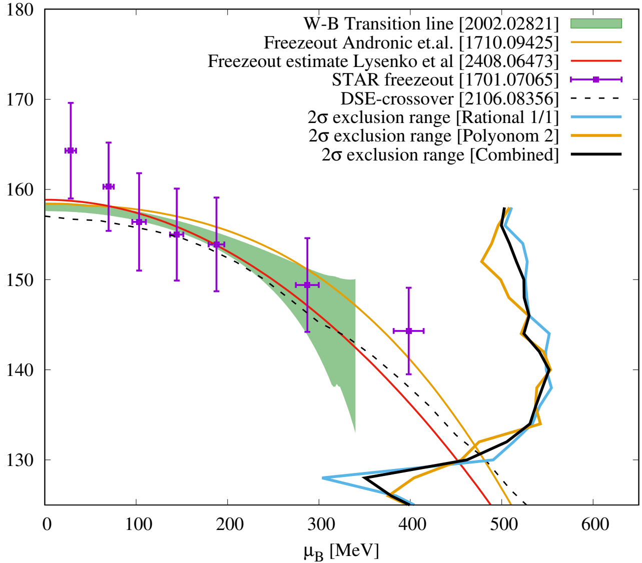


3. Pick a μ_B : $s(T_s, \mu_B) = s(T_0, 0)$

4. $\mu_B \geq \mu_{B,c}$ means multivalued s, that is $T_s(T_1) = T_s(T_2)$.

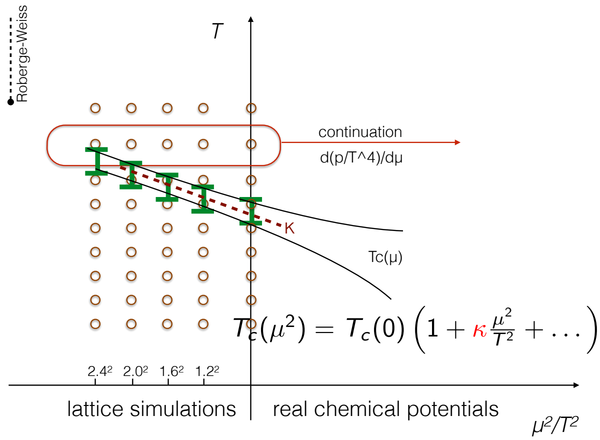


For how big μ_B can we show that $T_s(T_0)$ is strictly monotonic?
That will be the exclusion range.

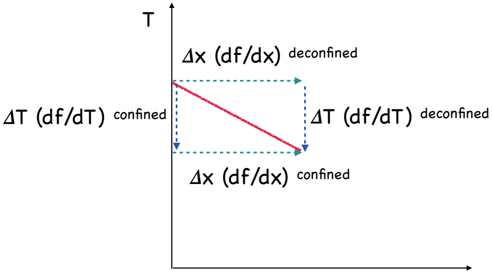

W-B Transition line [2002.02821]
Freezeout Andronic et.al. [1710.09425]
Freezeout estimate Lysenko et al [2408.06473]
STAR freezeout [1701.07065]
DSE-crossover [2106.08356]
2σ exclusion range [Rational 1/1]
2σ exclusion range [Polyonom 2]
2σ exclusion range [Combined]
180
170
160
150
140
130
0
100
200
300
400
500
600
μB [MeV]

QCD phase diagram

analytical continuation of lattice data



$$T_c(\mu^2) = T_c(0)\left(1 + \kappa \frac{\mu^2}{T^2} + \dots\right)$$



Free energy density f has no discontinuity

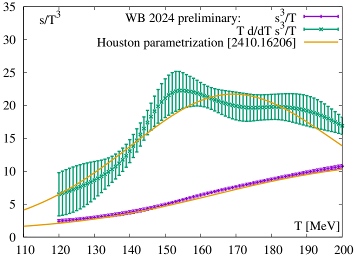
$$\Delta x \left.\frac{df}{dx}\right|_{\text{confined}} + \Delta T \left.\frac{df}{dT}\right|_{\text{confined}} = \Delta x \left.\frac{df}{dx}\right|_{\text{deconfined}} + \Delta T \left.\frac{df}{dT}\right|_{\text{deconfined}}$$

$$-\Delta s = \left.\frac{df}{dT}\right|_{\text{deconfined}} - \left.\frac{df}{dT}\right|_{\text{confined}} \quad ; \quad -\frac{1}{2}\Delta\chi = \left.\frac{df}{dx}\right|_{\text{deconfined}} - \left.\frac{df}{dx}\right|_{\text{confined}}$$

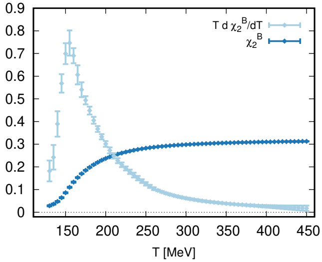
$$\Delta T \Delta s = -\Delta x \frac{1}{2}\Delta\chi$$

$$\kappa := \frac{1}{T} \cdot \frac{\Delta T}{\Delta x} = -\frac{\Delta\chi / T^4}{2\Delta s / T^3}$$

$$\frac{s}{T^3} \qquad T\frac{d}{dT}\frac{s}{T^3} \qquad \chi_2^B \qquad T\frac{d}{dT}\chi_2^B$$



[Wuppertal Budapest [preliminary]]



[Wuppertal Budapest [2102.06660]]

$$\kappa \;=\; -\frac{1}{2}\cdot\frac{\Delta\text{susceptibility}}{\Delta\text{entropy}} \rightarrow -\frac{1}{2}\cdot\frac{\frac{d\text{susceptibility}}{dT}}{\frac{d\text{entropy}}{dT}} \approx -\frac{1}{2}\cdot\frac{0.75}{22.3} \approx -0.017$$

$$P = P_R + P_I$$

Magenta color: C-even (real or imaginary).
Blue color: C-odd (real or imaginary)

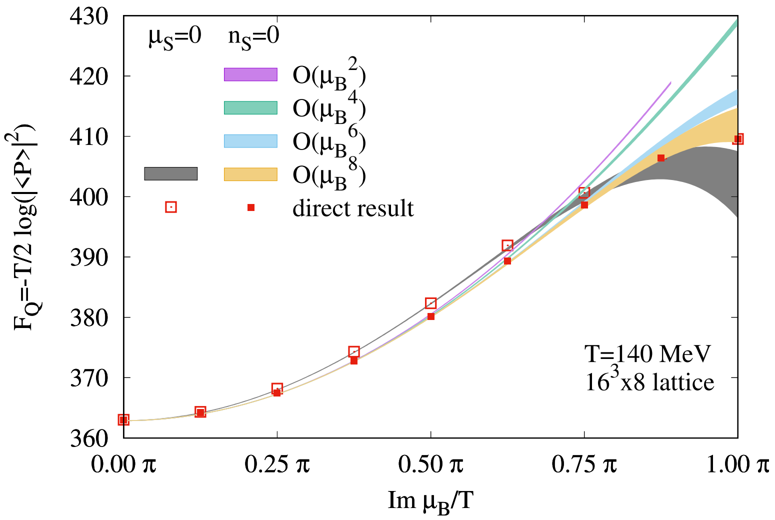
$$\begin{aligned}
\partial_j \langle P_R \rangle \big|_{\mu\equiv 0} &= 0\,, \\
\partial_j \langle P_I \rangle \big|_{\mu\equiv 0} &= \langle A_j P_I \rangle\,, \\
\partial_j \partial_k \langle P_R \rangle \big|_{\mu\equiv 0} &= \delta_{jk} \left[\langle B_j P_R \rangle - \langle B_j \rangle \langle P_R \rangle \right] \\
&\quad \langle A_j A_k P_R \rangle - \langle A_j A_k \rangle \langle P_R \rangle\,, \\
\partial_j \partial_k \langle P_I \rangle \big|_{\mu\equiv 0} &= 0\,.
\end{aligned}$$

If ∂_j is a **real chemical potential** derivative:

$$\langle P_I(\mu_B) \rangle = \underbrace{\langle A_j P_I(\mu_B = 0) \rangle}_{\text{real}} \mu_B + \ldots\,,$$

$\langle P_I \rangle$ was meant to be the imaginary part, but in fact,
it has a **real expectation value***, while being C-odd.*

Testing the strangeness neutral extrapolation:

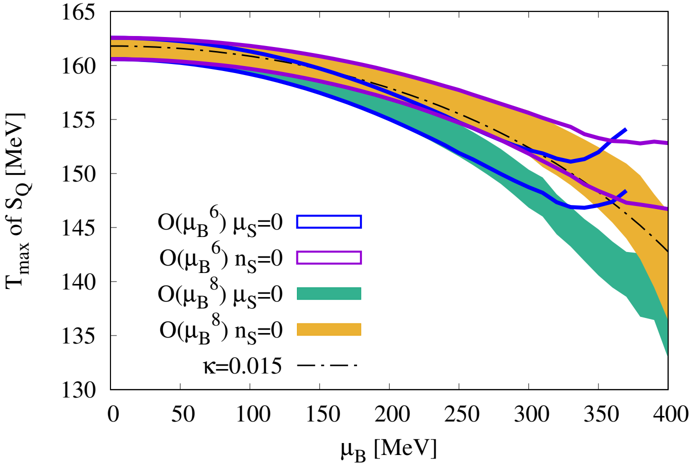


T_c extrapolated to eighth order

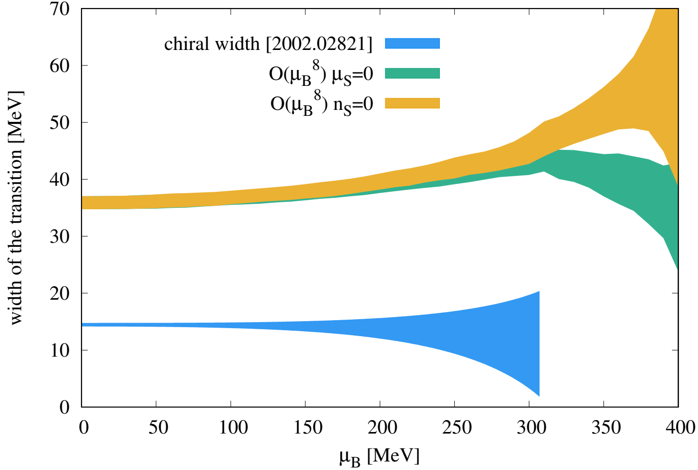
F_Q is extrapolated *($16^3 \times 8$ lattices)*.

Strangeness neutral ($n_S = 0$) and normal ($\mu_S = 0$) extrapolations differ.

The temperature of the S_Q-peak is calculated for each μ_B for both cases.



S_Q itself can be extrapolated *($16^3 \times 8$ lattices)*.
The **width** of the S_Q-peak is calculated for each μ_B.



Besides light baryons hyperons are also generated with $\mu_B > 0$: $\Longrightarrow \langle S \rangle < 0$

In experiment (at chemical freeze-out) $\langle S \rangle = 0$.

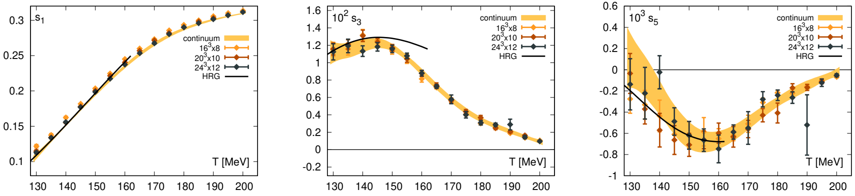
We achieve this by adding $\mu_S > 0$, this is T and μ_B dependent.

$$\mu_S^*(\mu_B, T) = s_1(T)\mu_B + s_3(T)\mu_B^3 + s_5(T)\mu_B^5 + \dots$$

One obtains $s_1(T)$, $s_3(T)$ and $s_5(T)$ from the standard Taylor coefficients

[HotQCD 1208.1220; 1701.04325]

Our recent continuum results [Wuppertal-Budapest 2312.07528]



Besides ligth baryons hyperons are also generated with $\mu_B > 0$: $\Longrightarrow \langle S \rangle < 0$
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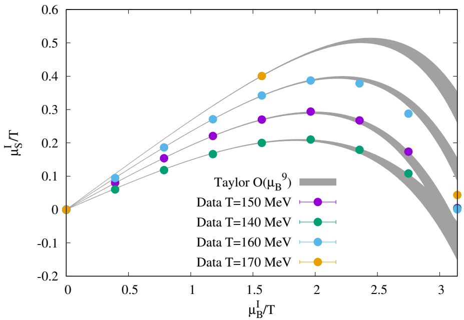
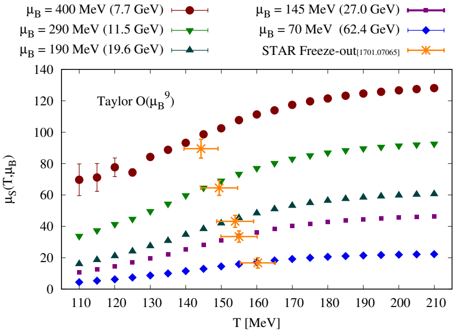
$$\mu_S^*(\mu_B, T) = s_1(T)\mu_B + s_3(T)\mu_B^3 + s_5(T)\mu_B^5 + s_7(T)\mu_B^7 + s_9(T)\mu_B^9 + \dots$$

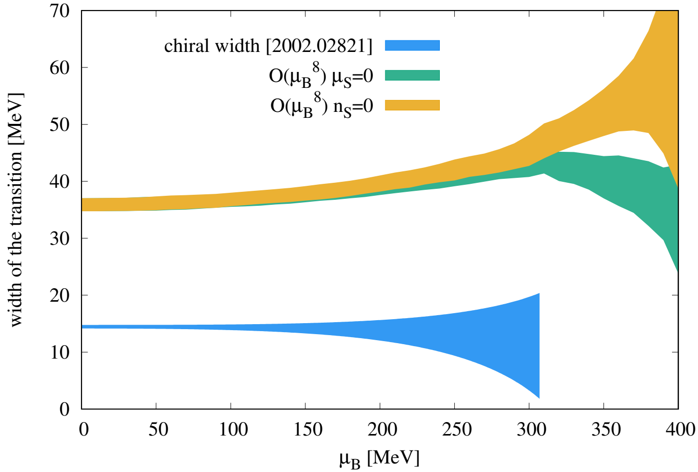




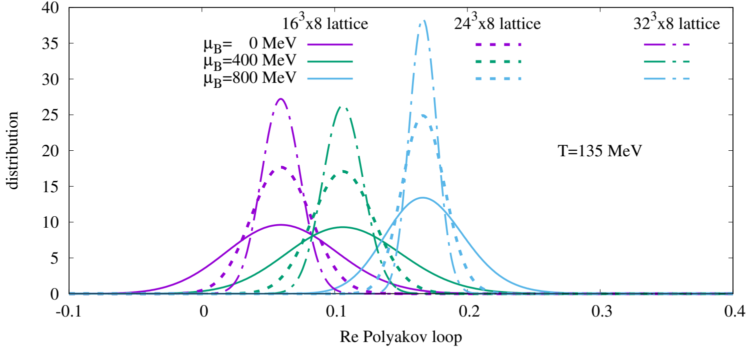
Wuppertal-Budapest preliminary data, $16^3 \times 8$ lattice.

Width of the transition extrapolated to eight order

S_Q itself can be extrapolated *($16^3 \times 8$ lattices)*.
The **width** of the S_Q-peak is calculated for each μ_B.



Polyakov loop histegrams at a fixed temperature

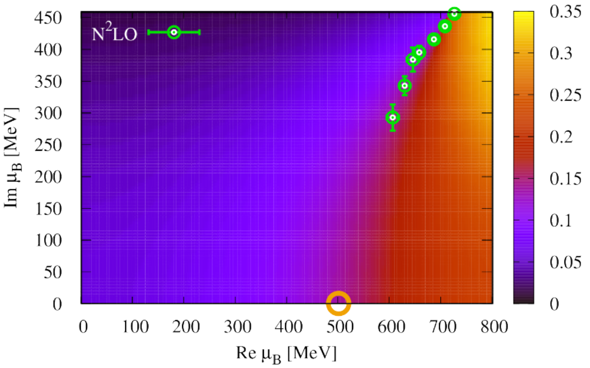


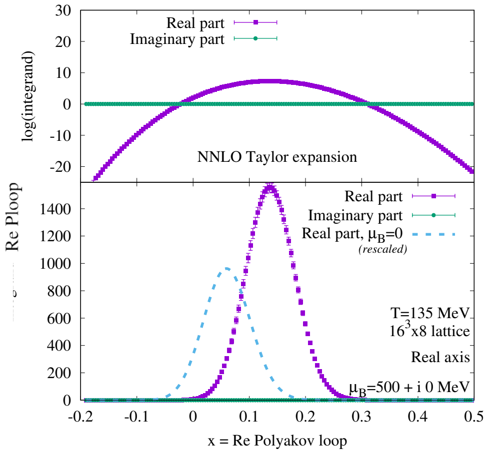
The density of states function was computed on a $16^3 \times 8$ *lattice*

$$\rho(x,\mu_B) = e^{(LT)^3\left(\chi_0(x)+\chi_2(x)\frac{hat\mu_B}{2!}+\dots\right)}$$

The complex μ_B plane at $T = 135$ MeV

$$Z(\mu) = \int_{-\infty}^{\infty} dx \exp\left((LT)^3 \left[\chi_0 + \frac{\hat{\mu}_B^2}{2!}\chi_2(x) + \frac{\hat{\mu}_B^4}{4!}\chi_4(x) + \frac{\hat{\mu}_B^6}{6!}\chi_6(x) \right] \right)$$





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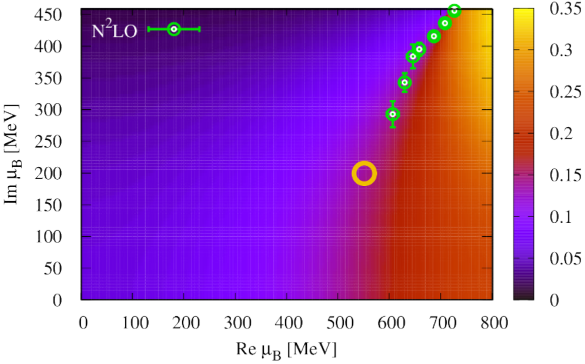

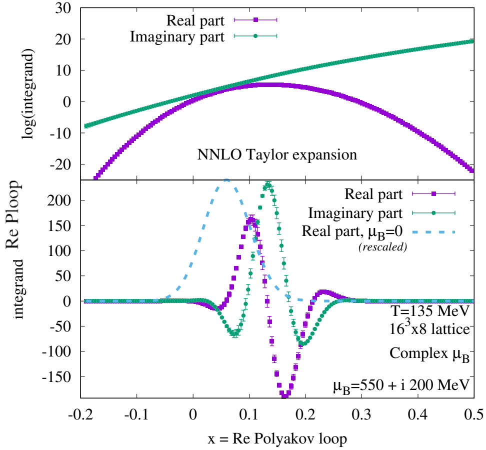

Yang-Lee edge in three volumes