



Quark numbers and percolation of electric center fluxes in QCD

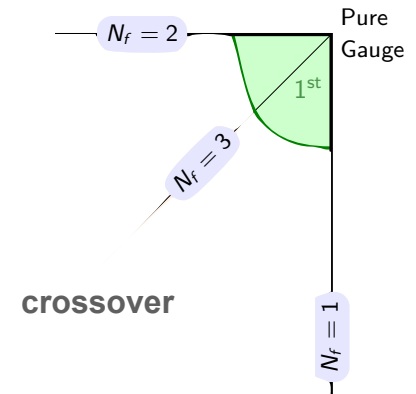
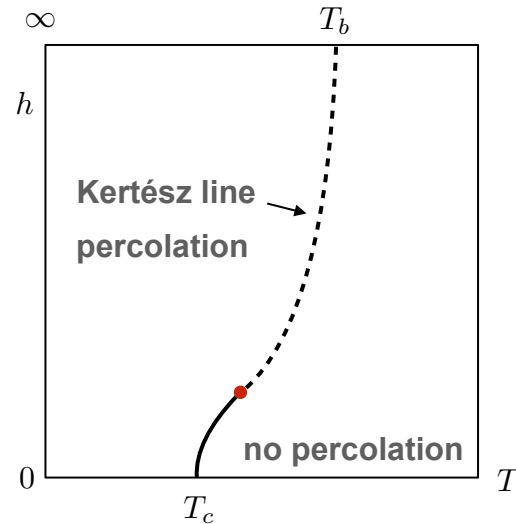
Bratislava, 9 May 2026

Lorenz von Smekal

**Milad Ghanbarpour, PhD thesis, JLU, July 2025
[doi:10.22029/jlupub-20047]**

PRD 106 (2022) 054513

Potts-Model Phase Diagram



Physica A 161 (1989) 58–62
North-Holland, Amsterdam

EXISTENCE OF WEAK SINGULARITIES WHEN GOING AROUND THE LIQUID–GAS CRITICAL POINT

János KERTÉSZ¹

*Institute for Theoretical Physics, University of Cologne, Zùlpicher Strasse 77,
D-5000 Kôln 41, Fed. Rep. Germany*

Received 27 June 1989

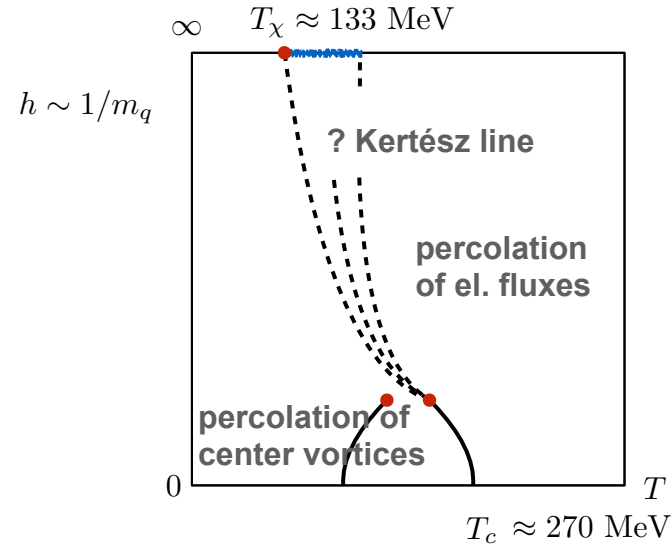
It has been known for more than a hundred years that one can drive a system around the liquid–gas critical point without any singular changes in the thermodynamic quantities. We argue that there are weak singularities in the droplet distribution corresponding to the disappearance of the surface tension of exponentially rare droplets off the coexistence curve. In the Coniglio–Klein–Swendsen–Wang droplet description of the Ising model such singularities occur in a natural way.

Potts model

Kertész completed his Diploma in Physics in
1976 from Eötvös University in Budapest

WIKIPEDIA 

Percolation in QCD



- **Percolation of electric center fluxes**
geometric deconfinement phase transition
- **where center-vortices form local clusters**
correlate with localized Dirac modes?

Baranka, Berta & Giordano, PRD 111 (2025) 074512

Quarks in a Finite Volume

- Why bother?
- consider canonical partition functions:

$$Z_c(T, V, N_q) = 0, \text{ for } N_q \not\equiv 0 \pmod{3}$$

→ Polyakov loop paradox

Kratochvila & de Forcrand, PRD 73 (2006) 114512

- follows from Roberge-Weiss symmetry

Roberge & Weiss, NPB 275 (1986) 734

- imaginary chemical potential: $\mu/T = i\theta$

fugacity expansion

→ Fourier series

period $2\pi/3$ →

$$Z^I(\theta) \equiv Z(T, V, i\theta T) = \sum_{N_q} e^{iN_q\theta} Z_c(T, V, N_q)$$

grand canonical at imaginary μ

canonical ensembles

→ only every 3rd Fourier coefficient $\neq 0$

Single Quark in Finite Volume

...we've just changed the temporal b.c.'s without changing the spatial ones!

- change spatial b.c.'s to account for Gauss' law

back up — pure $SU(N)$ gauge theory:

$(d+1)$ -dim spacetime

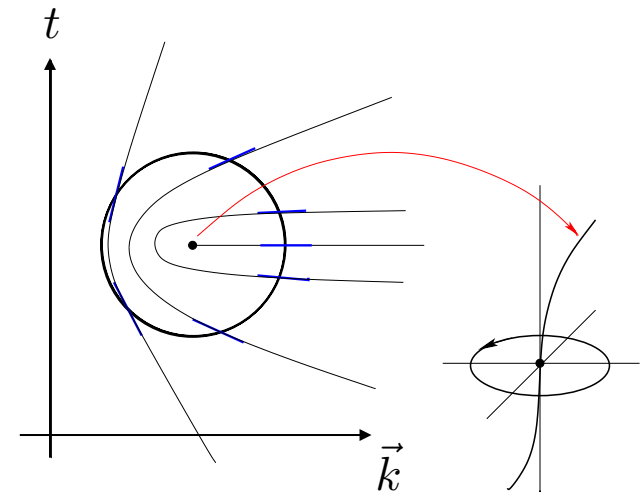
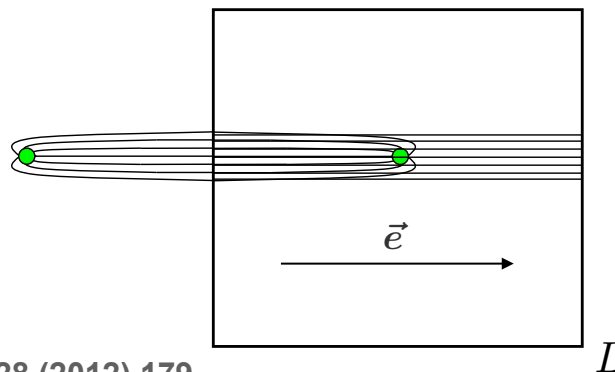
$$Z_e(\vec{e}) = \frac{1}{N^d} \sum_{\vec{k} \in \mathbb{Z}_N^d} e^{2\pi i \vec{e} \cdot \vec{k} / N} Z_k(\vec{k})$$

't Hooft's electric flux ensembles
(mirror charges)

't Hooft's twisted b.c.'s
(no of center vortices mod N)

$$\frac{Z_e(\vec{e})}{Z_e(0)} = \frac{1}{N} \left\langle \text{tr} (P_\Omega(\vec{x}) P_\Omega^\dagger(\vec{x} + \vec{e}L)) \right\rangle_{\text{no-flux}}$$

dual
↔



LvS, NPB (PS) 228 (2012) 179

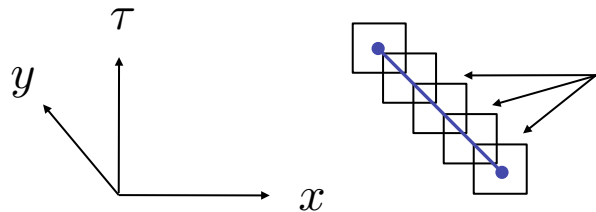
't Hooft's Twisted B.C.'s

- fix total # mod. N of center vortices through planes

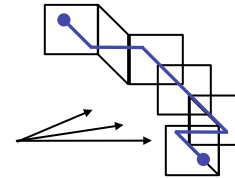
Kovacs & Tomboulis, PRL 85 (2000) 704

- implement on lattice

multiply plaquette couplings by non-trivial center element

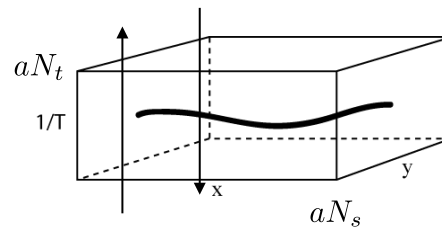


change link variables

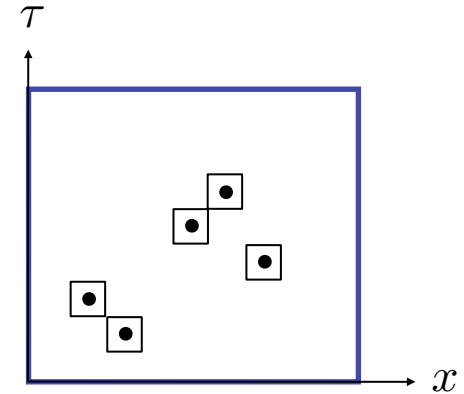
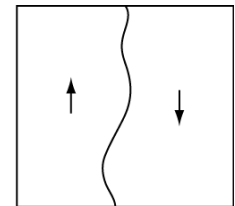


- for Polyakov loops

act as interfaces in spin model
dual to electric fluxes

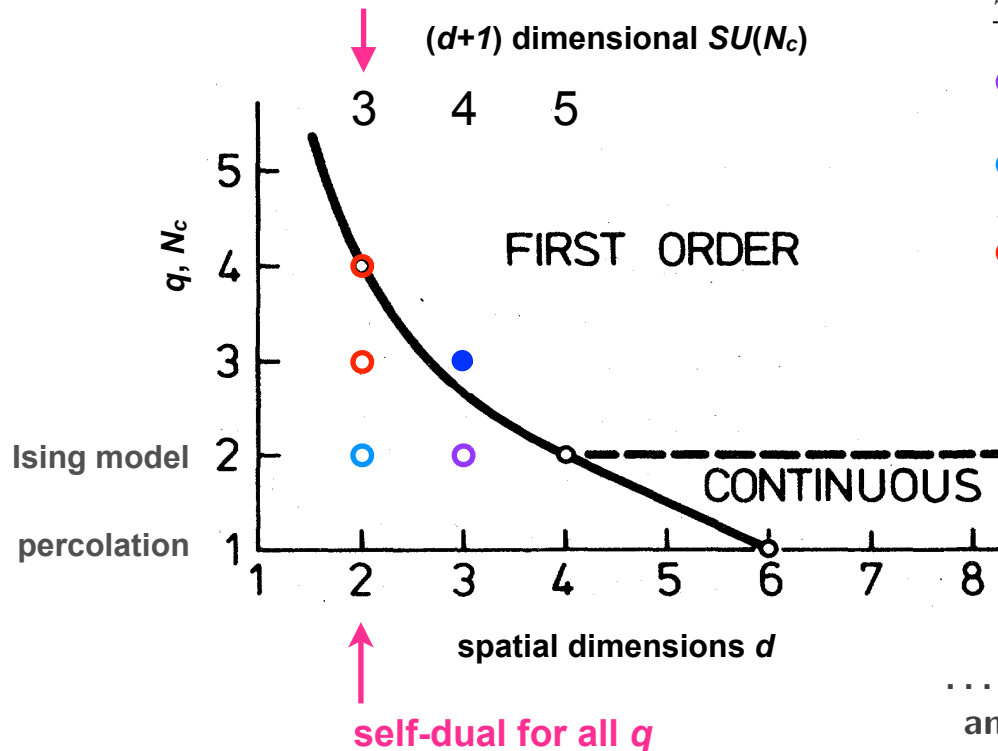


universality



• Potts model separatrix

F.Y. Wu, Rev. Mod. Phys. 54 (1982) 235



• QCD without quarks

$SU(N_c)$ with 2nd order transition:

- Ph. de Forcrand & LvS, PRD 66 (2002) 0111504
- S. Edwards & LvS, PLB 816 (2009) 484
- N. Strodthoff, S. Edwards & LvS, PoS (Lattice 2010) 288

... critical exponents, critical couplings and temperatures, universal amplitude ratios, finite-size scaling, interface tension...

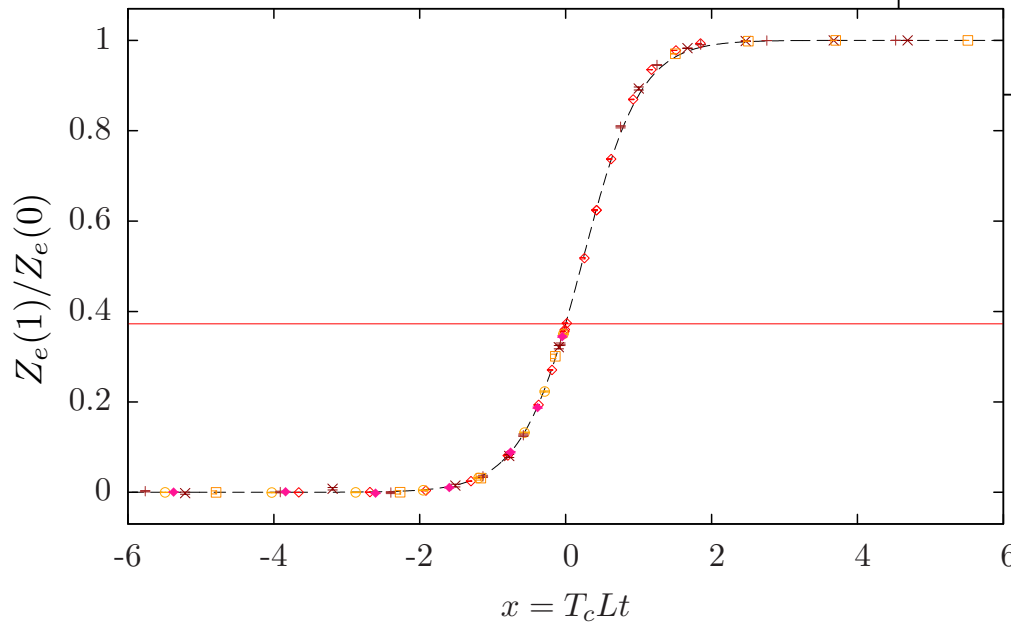
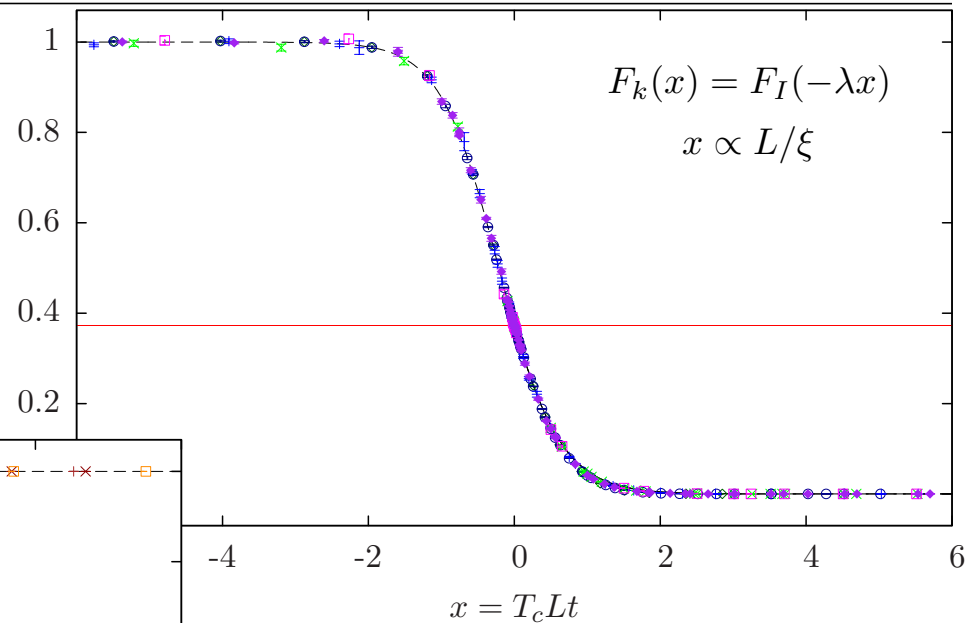
Center Vortices vs. Electric Fluxes

$F_I(x)$: Interface free energy in Ising model,
Wu et al., J. Phys. A: Math. Gen. 32 (1999) 4897

Center vortex:

$$F_k = -\ln(Z_k(1)/Z_k(0))$$

$$Z_k(1)/Z_k(0)$$



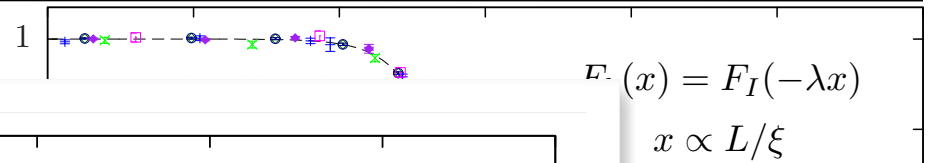
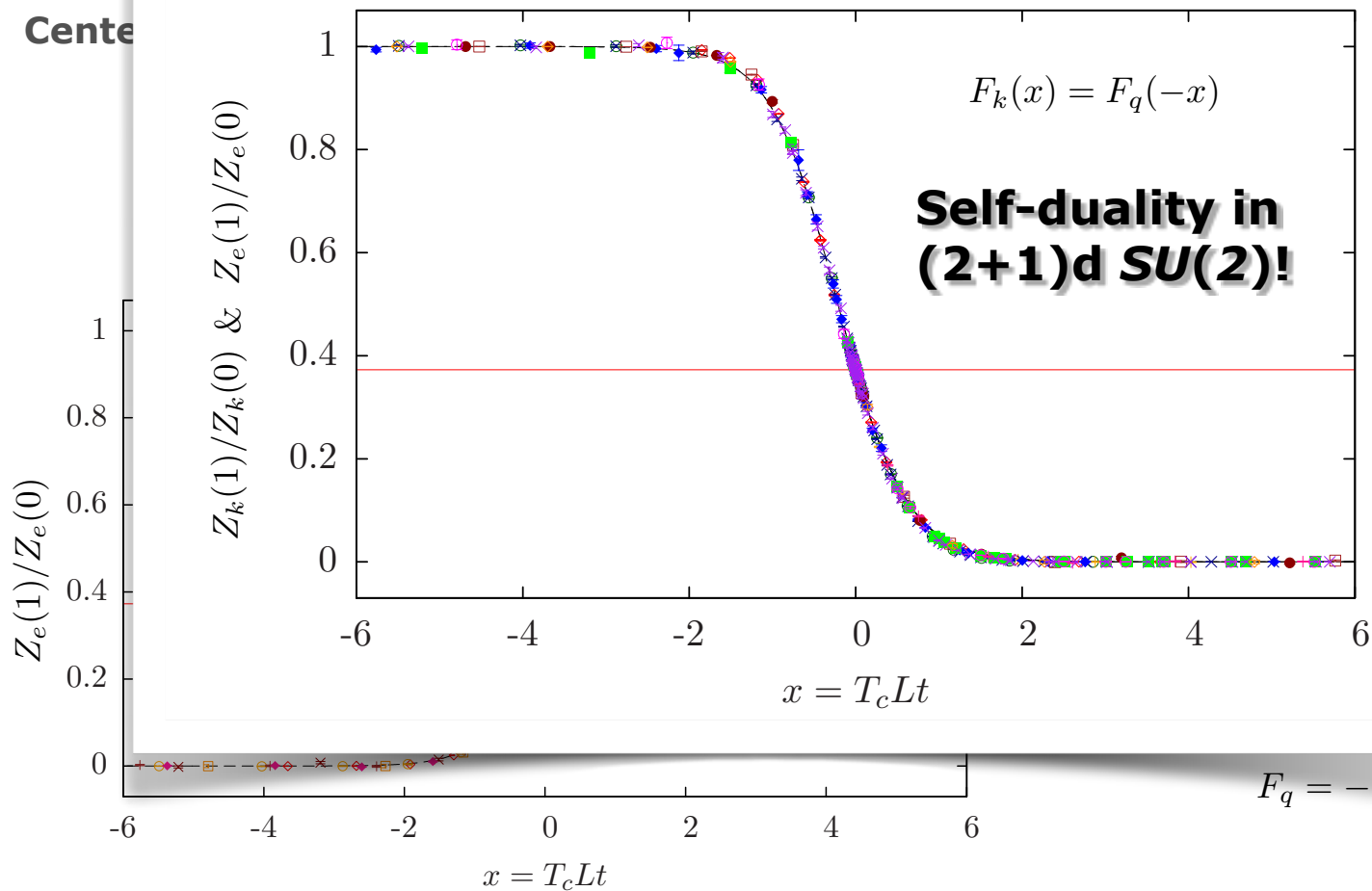
Electric flux:

$$F_q = -\ln(Z_e(1)/Z_e(0))$$

Center Vortices vs. Electric Fluxes

$F_I(x)$: Interface free energy in Ising model,
Wu et al., J. Phys. A: Math. Gen. 20 (1987) 4897

Center

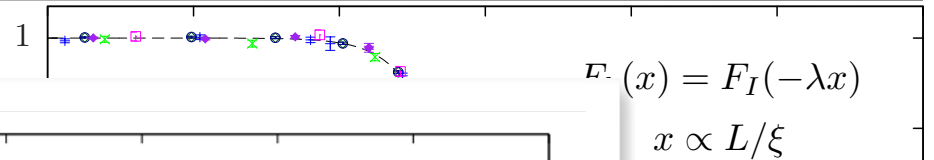


$$F_q = -\ln(Z_e(1)/Z_e(0))$$

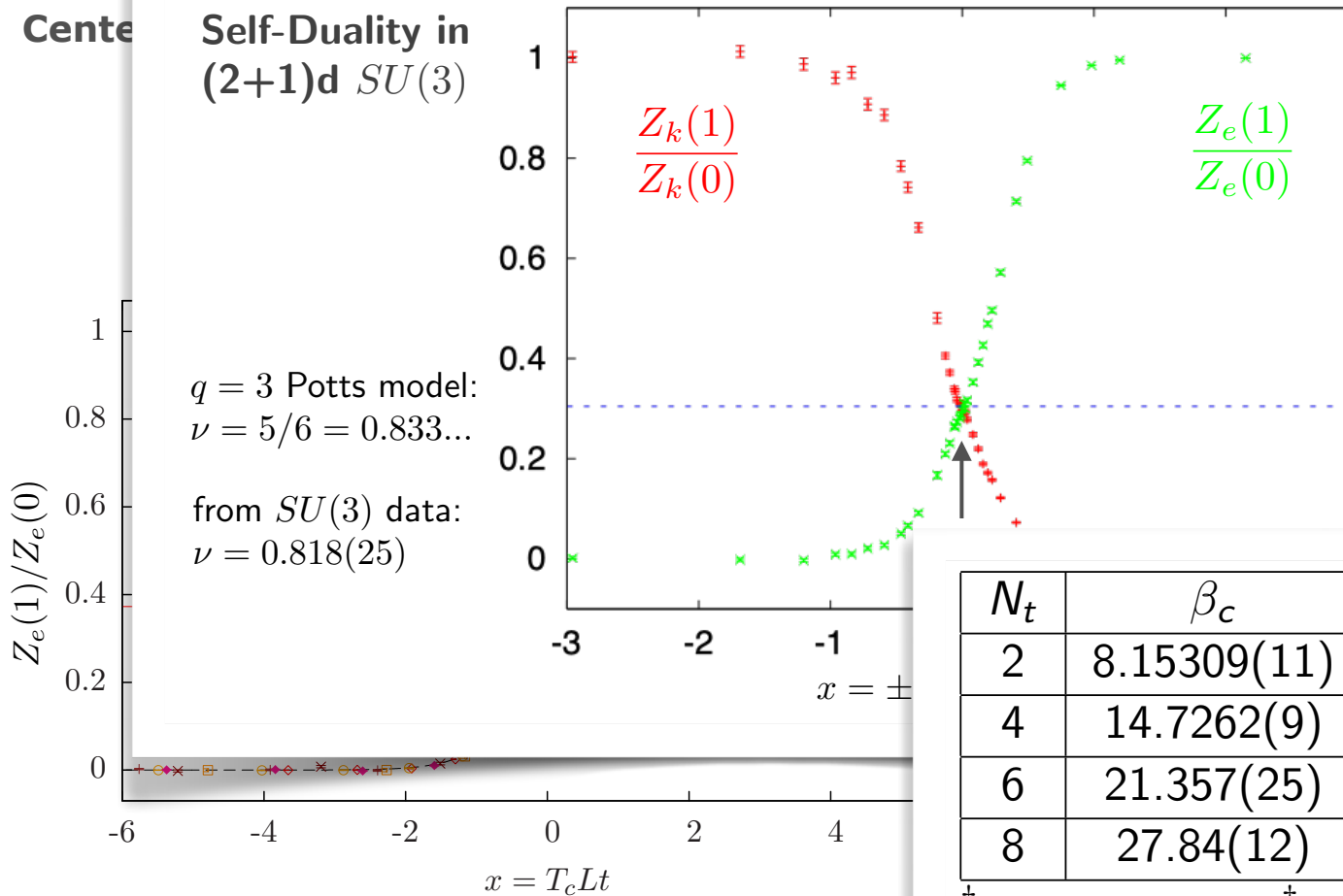
Center Vortices vs. Electric Fluxes

$F_I(x)$: Interface free energy in Ising model,

Wu et al., J. Phys. A: Math. Gen. 22 (1989) 4897



Center Self-Duality in (2+1)d $SU(3)$



N_t	β_c	Lit.
2	8.15309(11)	8.1489(31) [†]
4	14.7262(9)	14.717(17) [†]
6	21.357(25)	21.34(4) [‡]
8	27.84(12)	-

[†] [Liddle, Teper 2008]

[‡] [Engels et al. 1997]

Electric Fluxes in Full QCD

Kogut & Susskind, PRD 11 (1975) 395

- form of states:

$$|\psi\rangle = \sum \left(\underset{\substack{\uparrow \\ \text{set of spatial link variables}}}{f(U)} \otimes |\psi_F\rangle \right)$$

- implement Gauss law (physical States):

$$\underset{\substack{\uparrow \\ \text{generates spatial gauge transformations}}}{\hat{\rho}(\Omega)} |\psi\rangle = |\psi\rangle$$

- transform at single site:

$$\hat{\rho}(\Omega) \rightarrow \hat{\Pi}_i(\Omega) \prod_{j \sim i} \hat{\Pi}_{\langle i,j \rangle}(\Omega)$$

- generated by:

$$\underset{\text{color charges}}{\hat{Q}_i^a}$$

$$\underset{\text{color-electric fluxes}}{\hat{E}_{\langle i,j \rangle}^a}$$

$$(\hat{\Pi}_{\langle i,j \rangle}(\Omega) f)(U) =$$

$$\begin{cases} f(\{\dots, \Omega^\dagger U_{\langle i,j \rangle}, \dots\}) \\ f(\{\dots, U_{\langle j,i \rangle} \Omega, \dots\}) \end{cases}$$

Hilbert Space

- local Gauss law:

$$\hat{Q}_i^a = - \sum_{j \sim i} \hat{E}_{\langle i, j \rangle}^e \quad \text{in physical states}$$

but charges/fluxes not gauge invariant,
and don't commute

- restrict to Z_3 center:

$$\hat{Q}_i^z = \hat{\Pi}_i(z) \quad \hat{E}_{\langle i, j \rangle}^z = \hat{\Pi}_{\langle i, j \rangle}(z) \quad z \in \{1, e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i}\}$$

gauge invariant and commute

- local center charge and flux:

$$q, e \in \{0, 1, 2\}$$

$$\hat{Q}_i^z |q, e\rangle = z^{q_i} |q, e\rangle \quad \hat{E}_{\langle i, j \rangle}^z |q, e\rangle = z^{e_{\langle i, j \rangle}} |q, e\rangle$$

- decompose:

$$\mathcal{H} = \bigoplus_{\{q, e\}} \mathcal{H}_{\{q, e\}}$$

Physical States

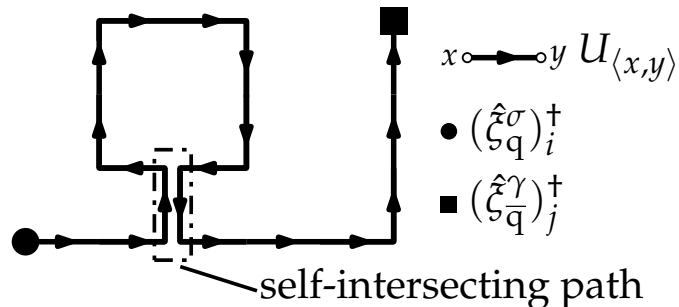
- local Z_3 Gauss law:

$$q_i + \sum_{j \sim i} e_{\langle i,j \rangle} = 0 \pmod{3}$$

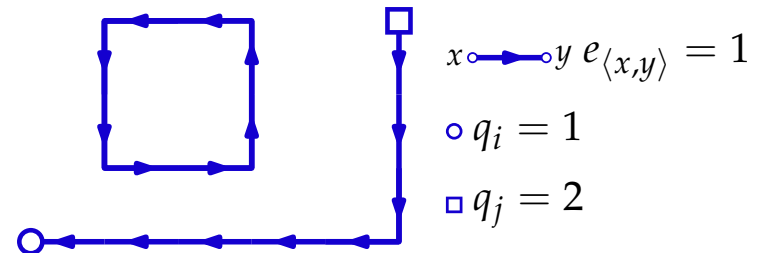
- physical center charge / flux states:

$$\mathcal{H}_{\text{phys}} = \bigoplus_{\{q,e\}_{\text{phys}}} \mathcal{H}_{\{q,e\}}$$

- mesonic state:



creation operator



center charge / flux configuration

- project onto these sectors:

$$\hat{P}_i(q) = \frac{1}{3} \sum_{z \in Z_3} z^{-q} \hat{Q}_i^z \qquad \hat{P}_{\langle i,j \rangle}(e) = \frac{1}{3} \sum_{z \in Z_3} z^{-e} \hat{E}_{\langle i,j \rangle}^z$$

- use Z_3 Gauss law:

$$\underbrace{\prod_{i \in V} \hat{Q}_i^z}_{=\hat{Q}_V^z} |\psi\rangle = \underbrace{\prod_{\langle i,j \rangle \in \mathcal{S}^*} \hat{E}_{\langle j,i \rangle}^z}_{=-\hat{\Phi}_{\mathcal{S}=\partial V}^z} |\psi\rangle$$

to implement charges via fluxes

- define projection operator

flux e through $\mathcal{S} = \partial V$

$$\hat{P}_{\mathcal{S}}(e) = \frac{1}{3} \sum_{z \in Z_3} z^{-e} \hat{\Phi}_{\mathcal{S}}^z$$

Mack, PLB 78 (1978) 263

Kijowski & Rudolph, J. Math. Phys. 43 (2002) 1796; *ibid.* 46 (2005) 032303

Transfer Operator

- partition function:

$$Z(\beta, \mu) = \text{Tr} \left(e^{\beta \mu \hat{N}_q} \hat{T}^{N_4} \hat{P}_0 \right)$$



project on gauge-invariant states
(with Gauss' law)

$$\hat{P}_0 |\psi\rangle = \int \mathcal{D}h \, \hat{q}(h) |\psi\rangle$$

- transfer operator:

$$\hat{T}(f(U) \otimes |\psi_F\rangle) = \int \mathcal{D}U' K(U, U') (f(U') \otimes |\psi_F\rangle)$$

- with kernel:

$$K(U, U') = T_F^\dagger(U) T_G^\dagger(U) S(U, U') T_G(U') T_F(U')$$

symmetric, Lüscher

$$K(U, U') = S(U, U') T_G^\dagger(U') T_G(U') \tilde{T}_F(U')$$

asymmetric, Milad

same PI representation of partition function

hermitian (Wilson fermion) Hamiltonian in time-continuum limit

Lüscher, Com. Math. Phys. 54 (1977) 283

Borgs & Seiler, Com. Math. Phys. 91 (1983) 329

Palumbo, NPB 645 (2002) 309

Mitrjushkin, NPB (PS) 119 (2003) 326

Transfer Operator

- apply center-flux operator:

$$(\hat{E}_{\langle i,j \rangle}^z K)(U, U') = S(U^z, U') T_G^\dagger(U') T_G(U') \tilde{T}_F(U')$$

$\uparrow$
 only acts on spatial link variables here,

$$U^z = \begin{cases} \{\dots, z^* U_{\langle i,j \rangle}, \dots\} \\ \{\dots, U_{\langle j,i \rangle} z, \dots\} \end{cases}$$

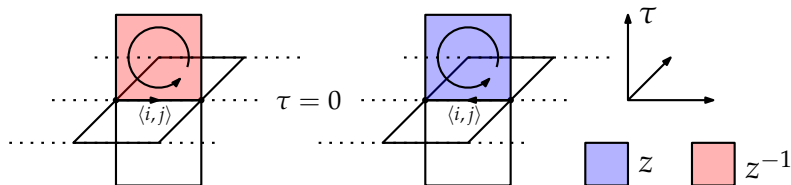
- EVs of center-flux configurations:

$$\left\langle \prod_{\langle i,j \rangle \in \mathcal{S}^*} \hat{E}_{\langle i,j \rangle}^z \right\rangle = \frac{1}{Z} \text{Tr} \left(\left[\prod_{\langle i,j \rangle \in \mathcal{S}^*} \hat{E}_{\langle i,j \rangle}^z \right] e^{\beta \mu \hat{N}_q} \hat{T}^{N_4} \hat{P}_0 \right)$$

single plaquette flip for $\langle \hat{E}_{\langle i,j \rangle}^z \rangle$

$$= \int \mathcal{D}[\dots] e^{-S_G^z(U, \{z\})} e^{-S_F(\bar{\psi}, \psi, U, \mu)}$$

$\uparrow$
 flip all temporal plaquettes $U_p \rightarrow z^* U_p$
 with spatial link in $\mathcal{S}^*$



$\langle i, j \rangle$: forward

backward link

Closed Center Vortex Sheets

- pure gauge theory

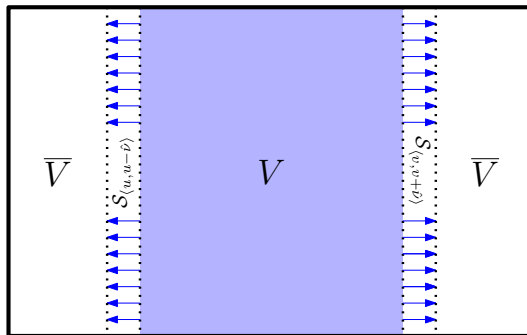
remove with variable transform

- heavy-dense limit of QCD

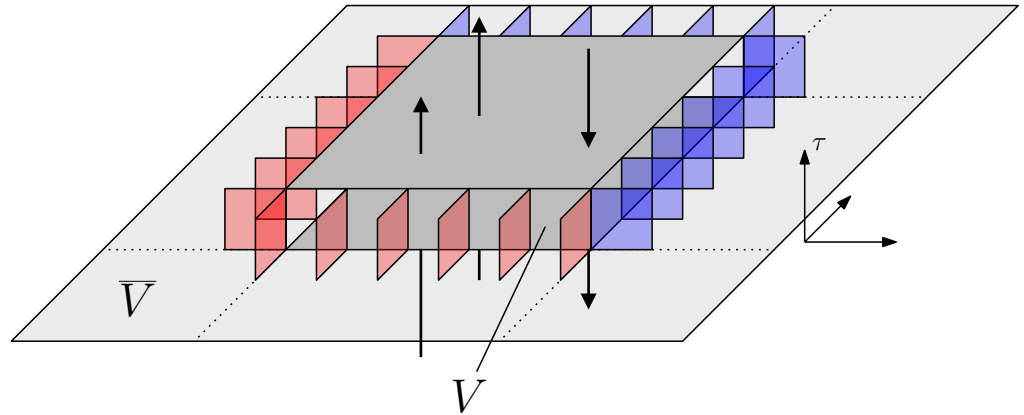
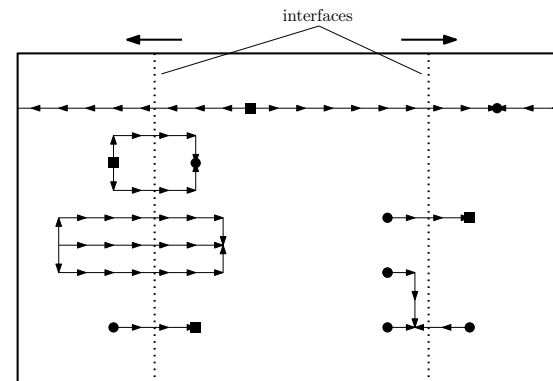
static fermion determinant

- Z_3 -Fourier transform over closed center vortex sheets

fix electric flux through $S = \partial V$



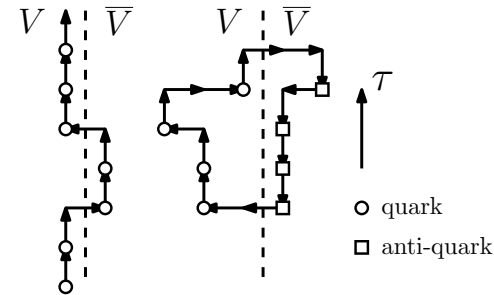
or net quark number mod. 3 inside



Quarks in a Finite Volume

- with arbitrary spatial hops

(anti-)quarks can hop in and out of V



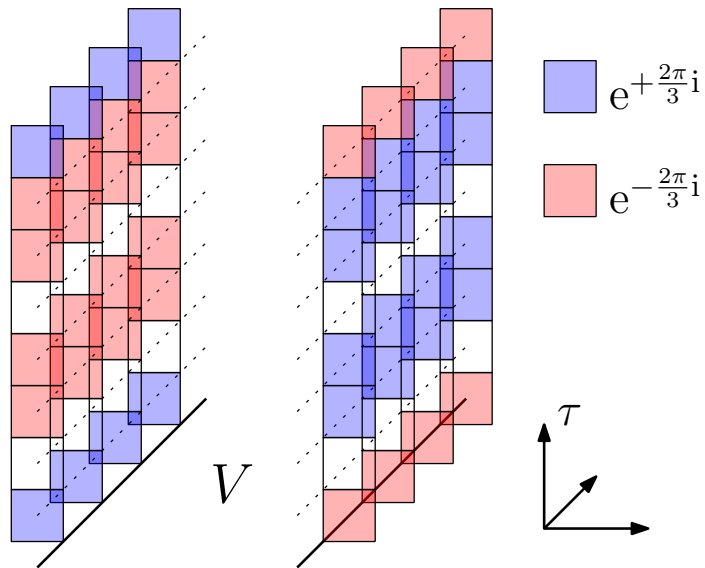
- introduce between *all* time slices

N_τ closed center-vortex sheets

- Z_3 -Fourier transforms

over N_τ closed center-vortex sheets

→ selective static membrane at $S = \partial V$
(only hadrons can pass)



- effective Polyakov-loop theory

(1 flavor Wilson)

$$Z_{\text{eff}} = \int \left(\prod_i dL_i J(L_i) Q(L_i) \right) \prod_{\langle i,j \rangle} (1 + 2\lambda \text{Re} L_i L_j^*)$$

Fromm, Langelage, Lottini, Philipsen, JHEP 01 (2012) 042
Langelage, Neuman, Philipsen, JHEP 09 (2014) 131

leading order hopping expansion
static fermion determinat $\rightarrow$ site factors

$$Q(L) = (1 + hL + h^2 L^* + h^3)^2 (1 + \bar{h}L^* + \bar{h}^2 L + \bar{h}^3)^2$$

where

$$h(\mu) = e^{(\mu-m)/T}$$

$$\bar{h}(\mu) = h(-\mu)$$

Pietri, Feo, Seiler, Stamatescu, PRD 76 (2007) 114501

- **for QCD at strong coupling**

with static fermion determinant

$$Z_{\text{eff}} = \mathcal{N} \sum_{\{z_i \in \mathbb{Z}_3\}} \exp \left\{ \sum_{\langle i, j \rangle} 2\gamma \operatorname{Re} z_i z_j^* \right\} \times \\ \left(\prod_i (1 + h z_i + h^2 z_i^* + h^3)^2 (1 + \bar{h} z_i^* + \bar{h}^2 z_i + \bar{h}^3)^2 \right)$$

with $\gamma = \frac{1}{3} \ln \left(\frac{1 + 2\lambda}{1 - \lambda} \right)$

- **Roberge-Weiss symmetric**

from global $\mathbb{Z}_3$ symmetry

$$Z_{\text{eff}}(T, \mu = i\theta T) \equiv Z_{\text{eff}}^I(\theta) = Z_{\text{eff}}^I(\theta + 2\pi/3)$$

Equivalent Flux-Tube Model

- flux-tube model representation (dual)

Ghanbarpour, LvS, PRD 106 (2022) 054513

$$Z_{\text{eff}}(T, \mu) = \sum_{\{n, l\}_{\text{phys}}} \exp \left\{ -\beta \left(H(n, l) - \mu \sum_i q_i \right) \right\}$$

analogous to:

Patel, NPB 243 (1984) 411

Bernard, DeGrand, DeTar, Gottlieb, Krasnitz, Sugar, Toussain, PRD 49 (1994) 6051

Condella & DeTar, PRD 61(2000) 074023

here with:

$$H(n, l) = \sum_{\langle i, j \rangle} \overset{\text{string tension}}{\sigma} |l_{\langle i, j \rangle}| + \sum_{i, s} m(n_{i, s} + \bar{n}_{i, s})$$

fluxes represented by link variables: $l_{\langle i, j \rangle} \in \{-1, 0, 1\}$

(anti-)quark occupation numbers: $n_{i, s} \in \{0, \dots, 3\}$ and $\bar{n}_{i, s} \in \{0, \dots, 3\}$ spin $s = \{\uparrow, \downarrow\}$

- Z₃-Gauss' law:**

(Poisson equation)

$$\sum_{j \sim i} l_{\langle i, j \rangle} - \sum_s (n_{i, s} - \bar{n}_{i, s}) = 0 \pmod{3}$$

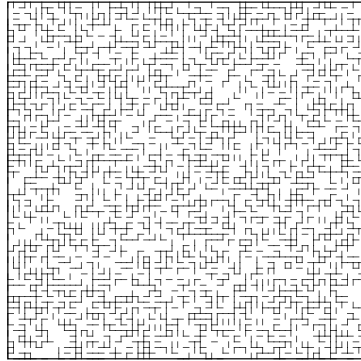
flux from volume
around site i

$$\underbrace{\sum_{j \sim i} l_{\langle i, j \rangle}}_{\phi_i} = \underbrace{\sum_s (n_{i, s} - \bar{n}_{i, s})}_{q_i \pmod{3}}$$

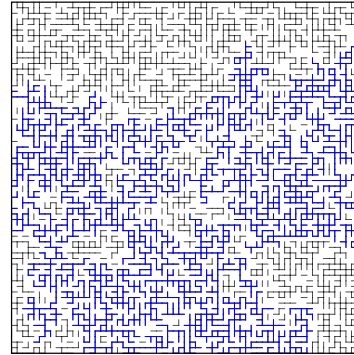
net-quark number modulo 3

Bond Percolation

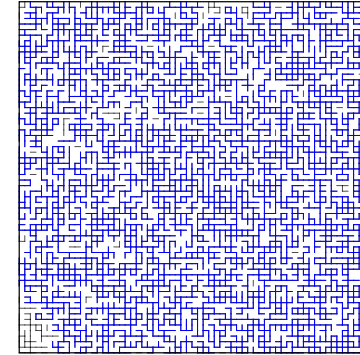
- place bonds randomly:



(a) $p = 0.4$



(b) $p = p_c = 0.5$



(c) $p = 0.6$

- find spanning cluster:

with probability

$$R_1(p, N) = \phi(A(p - p_c)N^{1/\nu})$$

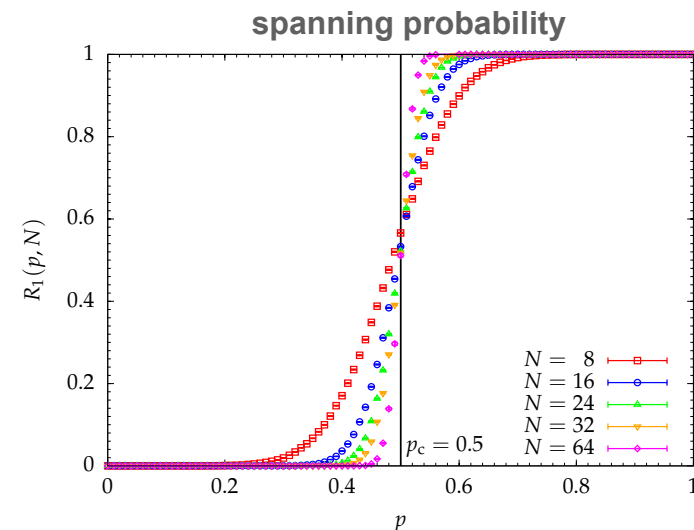
in two dimensions

$$p_c = 1/2 \quad \nu = 4/3$$

in three dimensions

$$p_c = 0.24881182(10) \quad \nu^{-1} = 1.1410(15)$$

Wang, Zhou, Zang et al., PRE 87 (2013) 052107



- **expectation value:**

electric center-flux through link $\langle i, j \rangle$

$$\langle \hat{E}_{\langle i, j \rangle}^z \rangle = \frac{1}{Z} \text{Tr} \left(\hat{E}_{\langle i, j \rangle}^z e^{\beta \mu \hat{N}_q} \hat{T}^{N_4} \hat{P}_0 \right)$$

- **probability:**

of obtaining value $e \in \{0, 1, 2\}$

$$\begin{aligned} p(e_{\langle i, j \rangle}) &= \langle \hat{P}_{\langle i, j \rangle}(e) \rangle = \frac{1}{3} \sum_{z \in Z_3} z^{-e} \langle \hat{E}_{\langle i, j \rangle}^z \rangle \\ &= \frac{1}{3} \sum_{z \in Z_3} z^{-e_{\langle i, j \rangle}} \left\langle e^{\frac{2}{g^2} \text{ReTr}([z^* - 1]U_p)} \right\rangle \end{aligned}$$

- **bond probability:**

$$p_b = 1 - p(e_{\langle i, j \rangle} = 0)$$

$$= \frac{2}{3} \left\langle 1 - \cosh \left(\frac{\sqrt{3}}{g^2} \text{ImTr } U_p \right) e^{-\frac{3}{g^2} \text{ReTr } U_p} \right\rangle$$

Bond Probability in QCD

- strong-coupling limit:

$$p_b \rightarrow 0$$

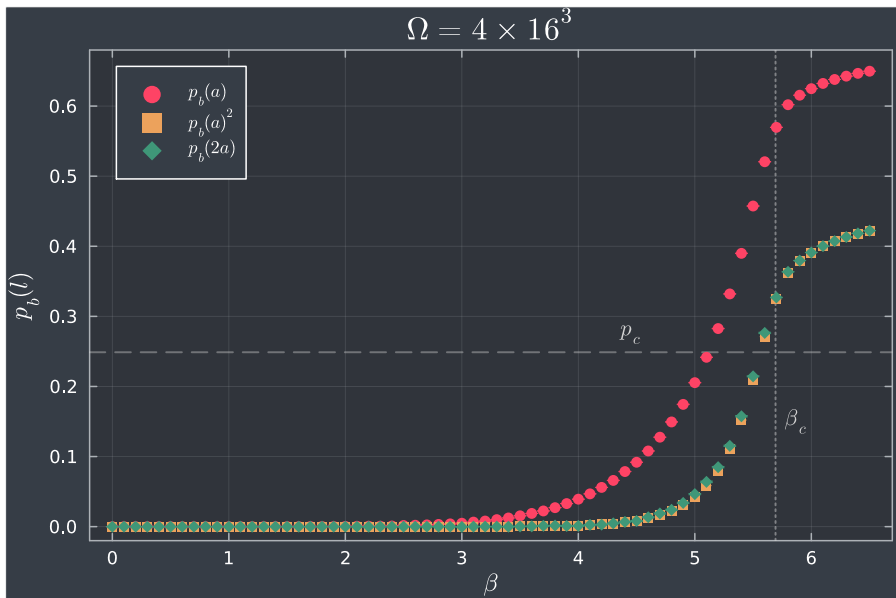
$< p_c$, never have percolation, confinement

- at weak coupling, high T :

$$p_b \rightarrow \frac{N_c - 1}{N_c} = \begin{cases} 1/2, & N_c = 2 \\ 2/3, & N_c = 3 \\ 1, & N_c \rightarrow \infty \end{cases}$$

- bond probability in pure SU(3):

asymptotically larger than p_c
in all cases, percolating electric fluxes
deconfinement



- spanning probability:

$$R_1(T, \mu, L) = \sum_{\{q,e\} \in \mathcal{R}_1} \frac{1}{Z} \text{Tr} \left(\hat{P}_{\{q,e\}} e^{\beta \mu \hat{N}_q} \hat{T}^{N_4} \hat{P}_0 \right)$$

Percolation in Potts Model

- **q-state Potts, Boltzmann factor:**

$$\omega(\{s, b\}) = \prod_{\langle i, j \rangle} (e^{-K} \delta_{b_{\langle i, j \rangle}, 0} + (1 - e^{-K}) \delta_{b_{\langle i, j \rangle}, 1} \delta_{s_i, s_j}) \prod_i e^{h \delta_{s_i, 0}}$$

↑ ↑
site-bond representation

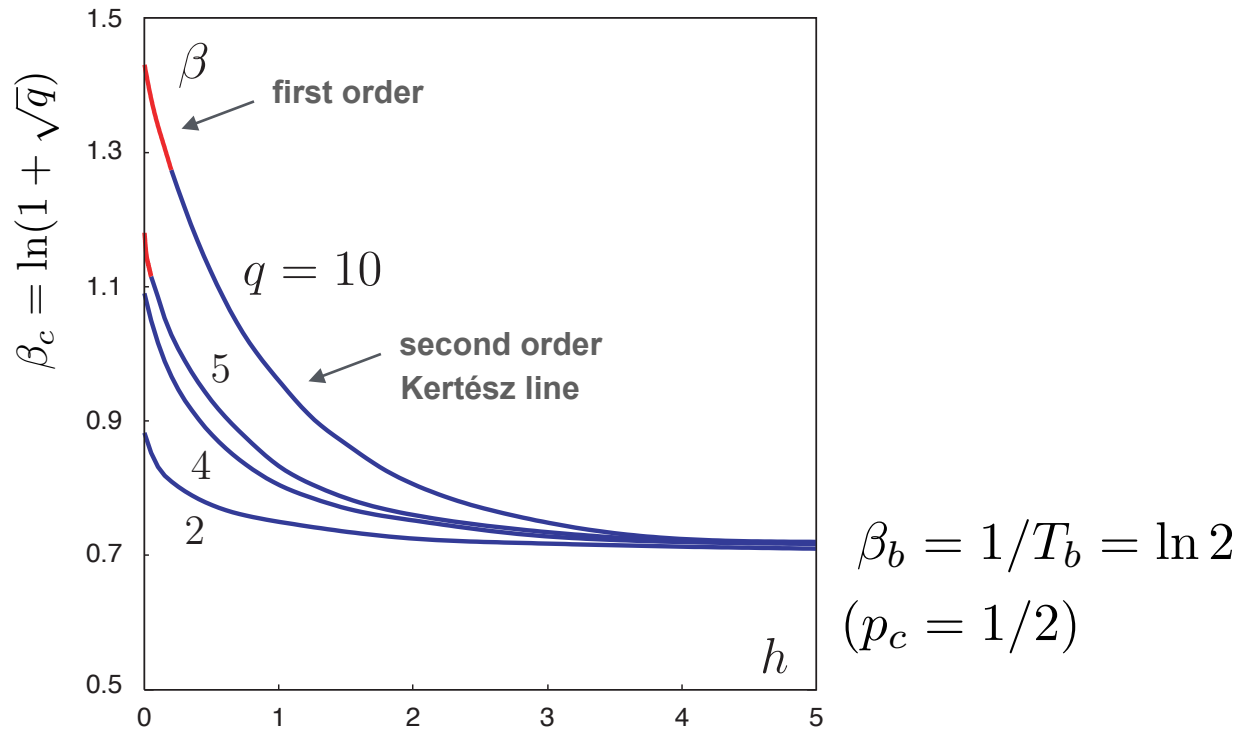
Edwards & Sokal, PRD 38 (1988) 2009

- **place bond:** $b_{\langle i, j \rangle} \in \{0, 1\}$ with probability $1 - e^{-K}$
between like nearest-neighbor spins $s_i \in \{0, 1, \dots, q-1\}$
- **infinite external field:** $h \rightarrow \infty \rightsquigarrow$ bond percolation
with bond probability $p = 1 - e^{-K}$, $K = J/T$ controlled by temperature
- **vanishing external field:** $h \rightarrow 0$,
if $p = p_c$ at $T = T_b > T_c \rightsquigarrow$ bond percolation in ordered phase below T_c
lose at Curie temperature T_c

Percolation in Potts Model

- q -state Potts, 2 dimensions:

Blanchard, Gandolfo, Laanait, Ruiz,
Satz, J. Phys. A 41 (2008) 085001



- spanning probability:

$$R(T, \mu, L) = \frac{1}{Z_{\text{flux}}} \sum_{\{n, l\} \in \mathcal{R}} \exp \{ -\beta (H(n, l) - \mu q) \}$$

set of percolating configs $\mathcal{R}$: contain at least one cluster of bond configurations spanning the entire volume in at least one direction

- simulate with worm algorithm

Prokof'ev & Svistunov, PRL 87 (2001) 160601
Korzec & Vierhaus, 2011, CPC 182 (2011) 1477
Delgado, Evertz, Gatteringer, CPC 183 (2012) 1920
Rindlisbacher, Akerlund, de Forcrand, NPB (2016) 542

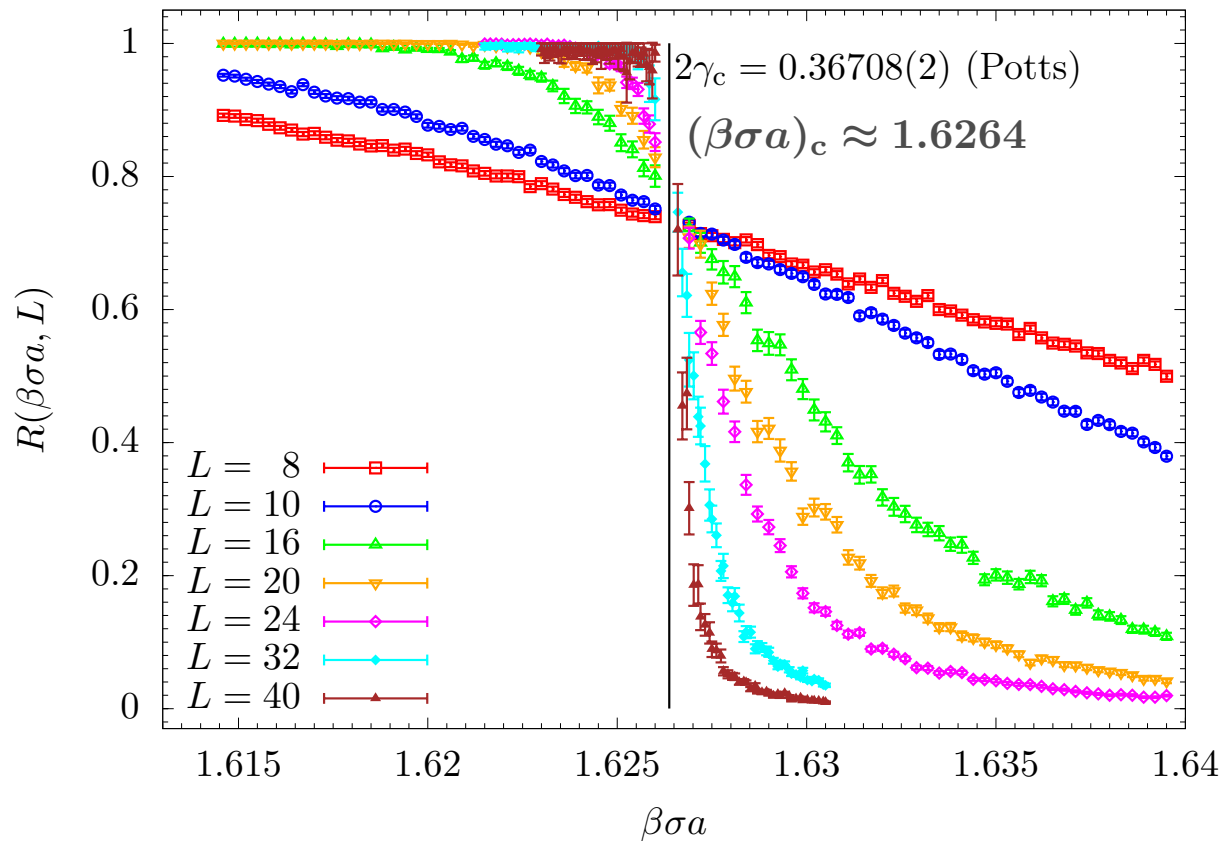
- measure with fully-dynamic connectivity algorithm

Holm, Lichtenberg, Thorup, J. ACM 48 (2001) 723
Alexandru, Bergner, Schaich, Wenger, PRD 97 (2018) 114503

Percolation in Flux-Tube Model

- infinitely heavy quarks
 Z_3 -Potts (1st order transition)

$$m \rightarrow \infty, \mu = 0$$

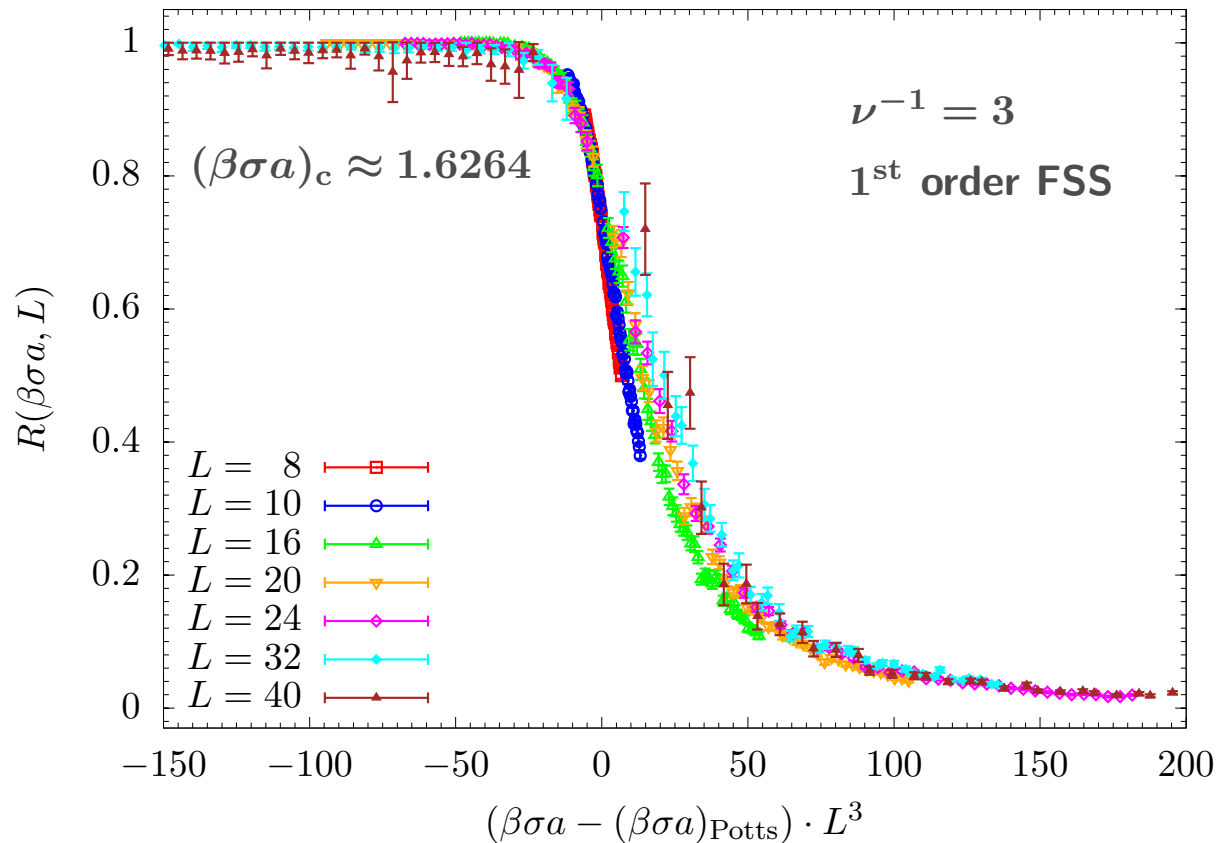


Percolation in Flux-Tube Model

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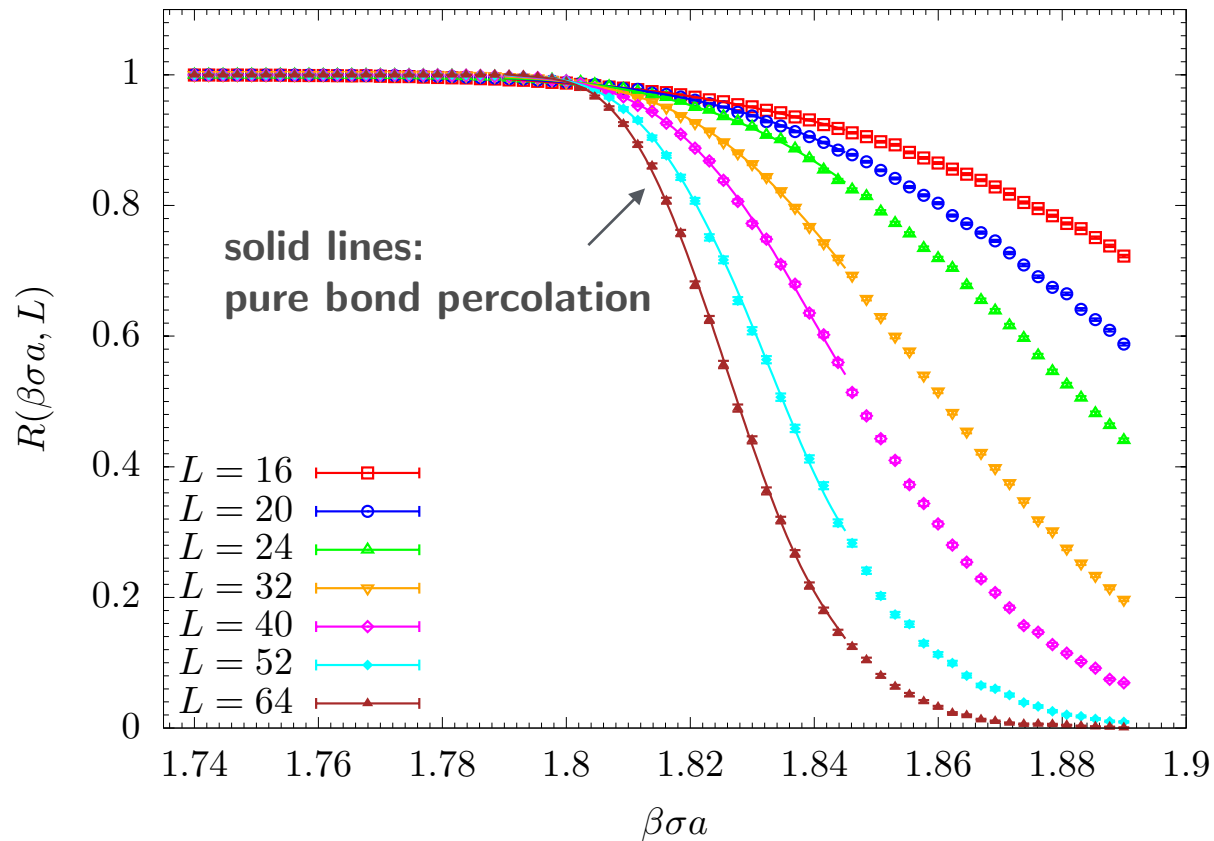


Percolation in Flux-Tube Model

- massless limit

bond percolation (2nd order)

$$m = 0, \mu = 0$$



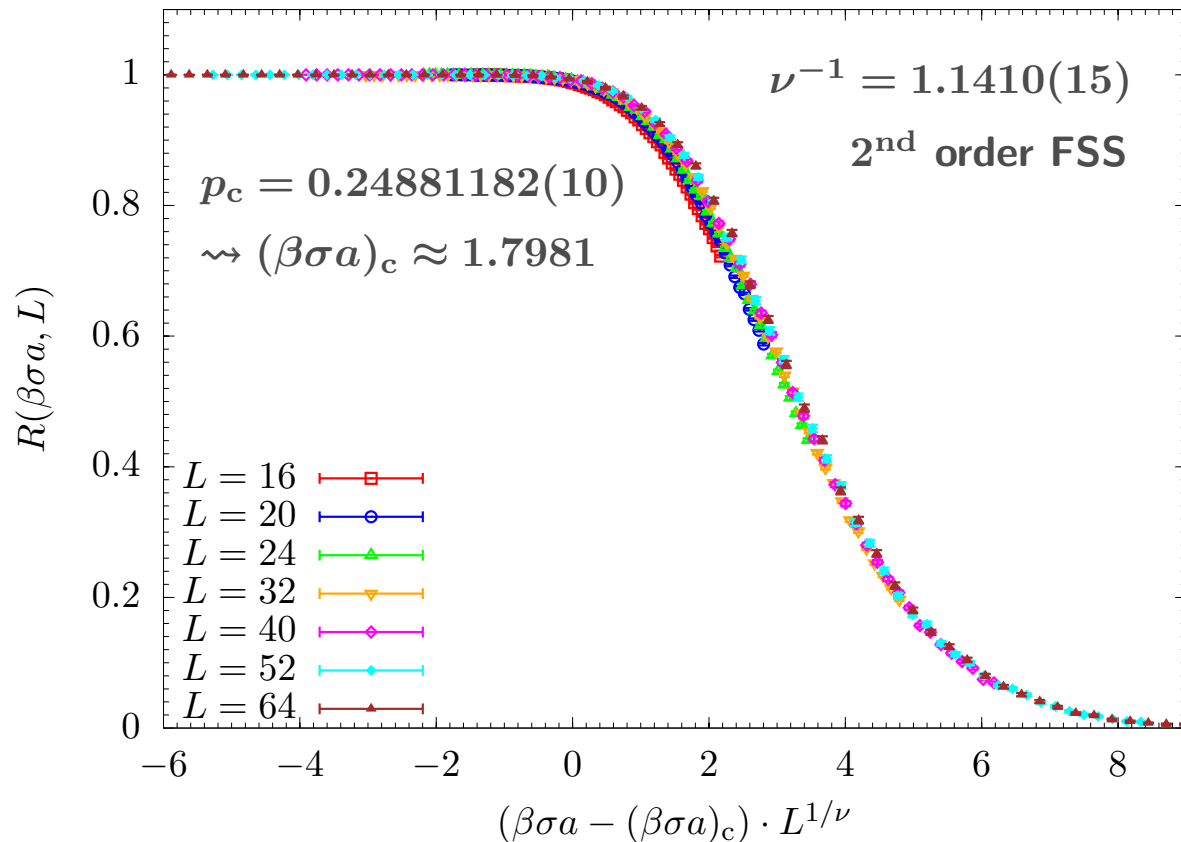
Percolation in Flux-Tube Model

- **massless limit**

Wang, Zhou, Zhang et al., PRE 87 (2013) 052107

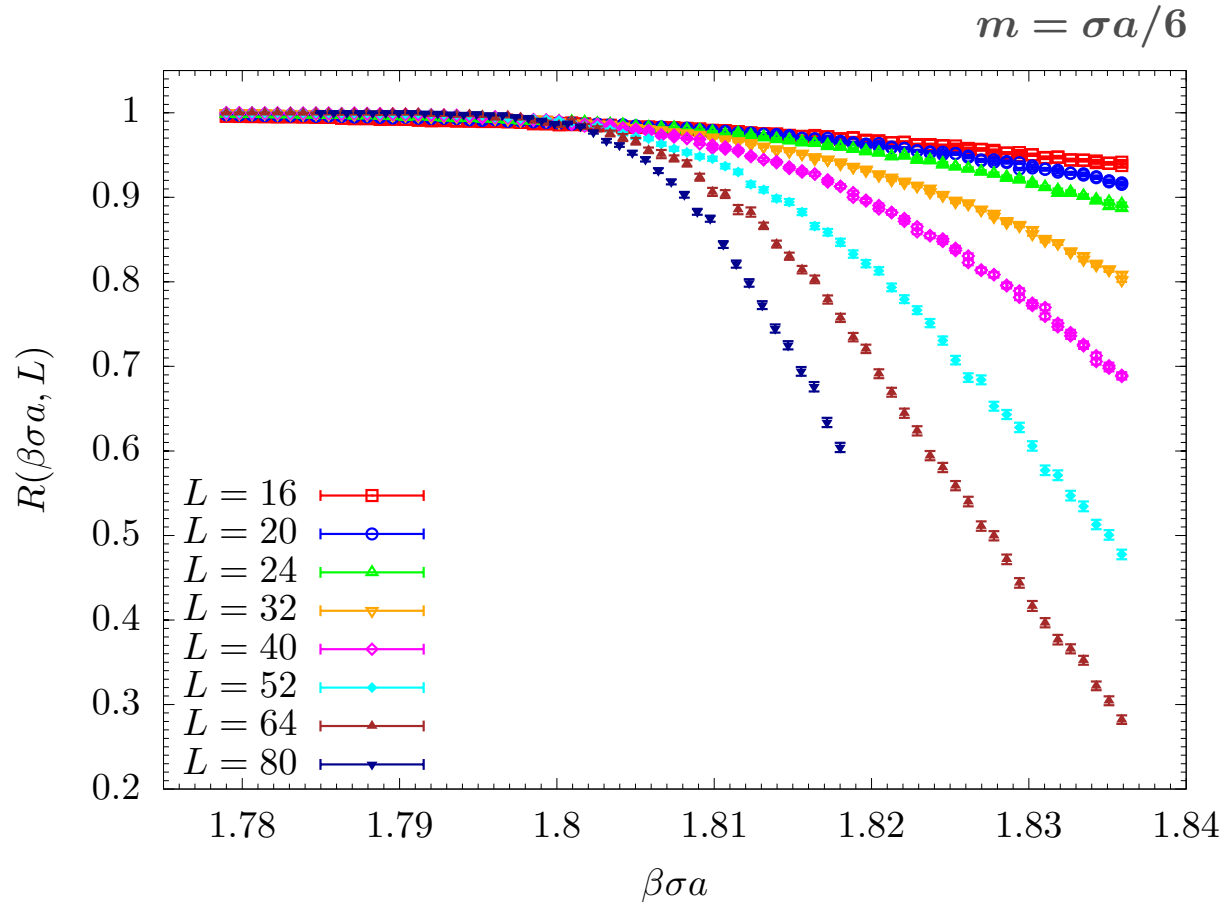
bond percolation (2nd order)

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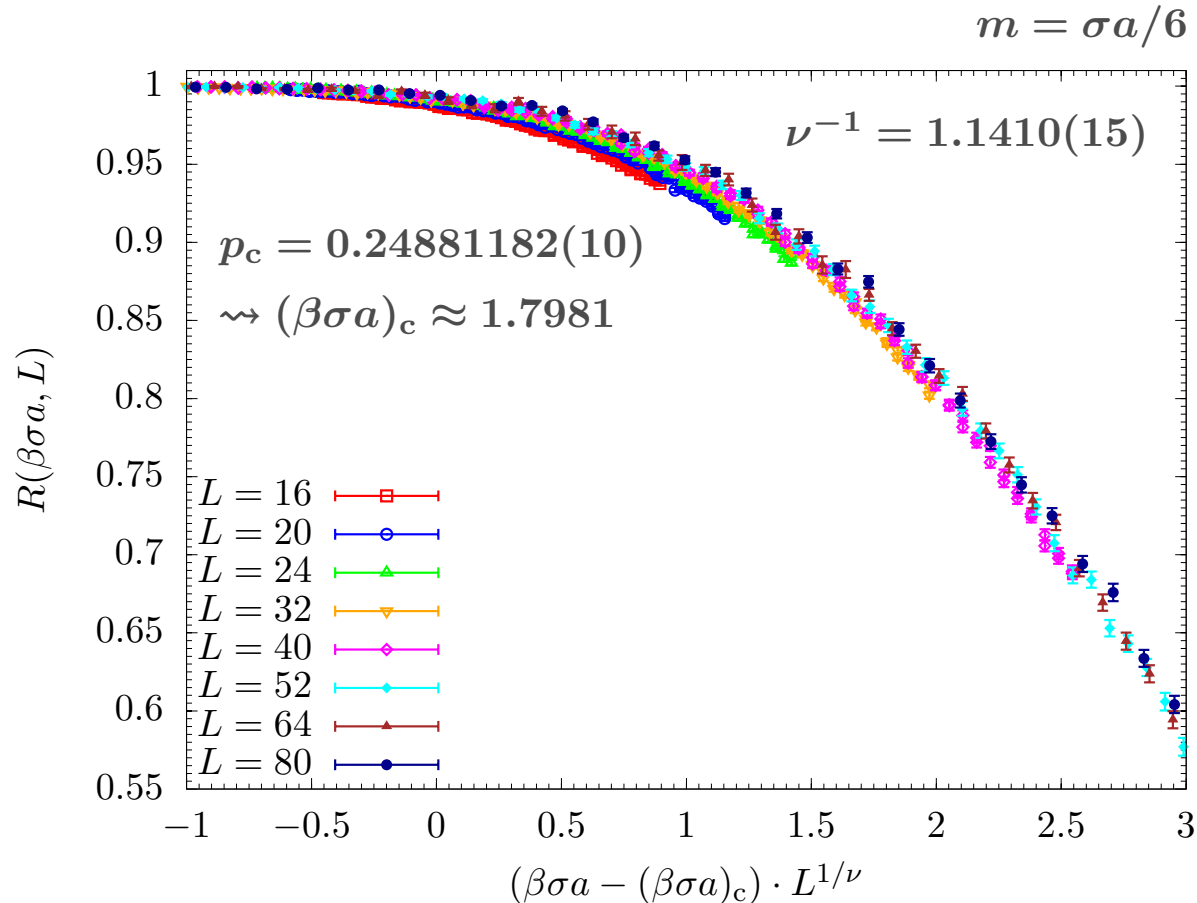
Percolation in Flux-Tube Model

- fairly light quarks
smooth Z_3 -Potts crossover



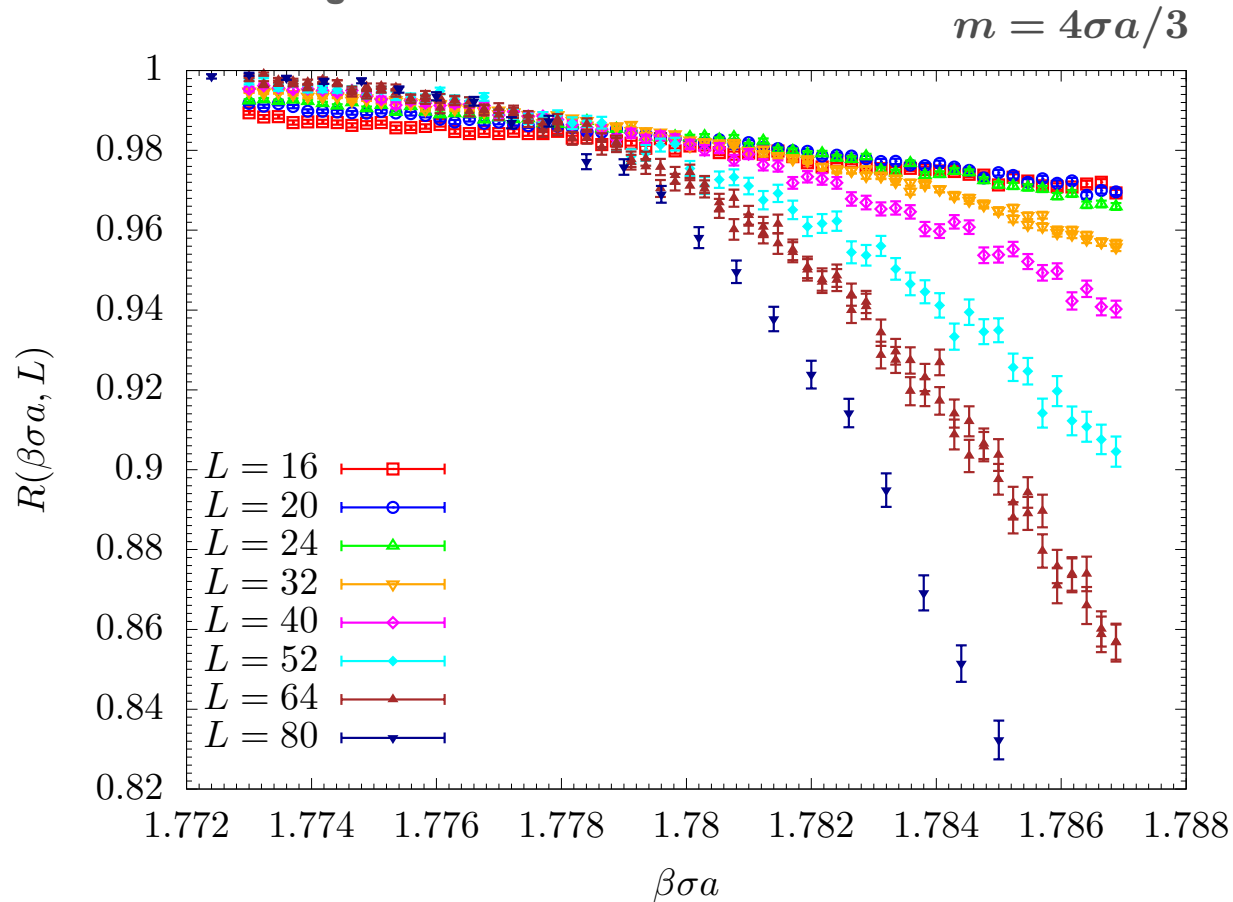
Percolation in Flux-Tube Model

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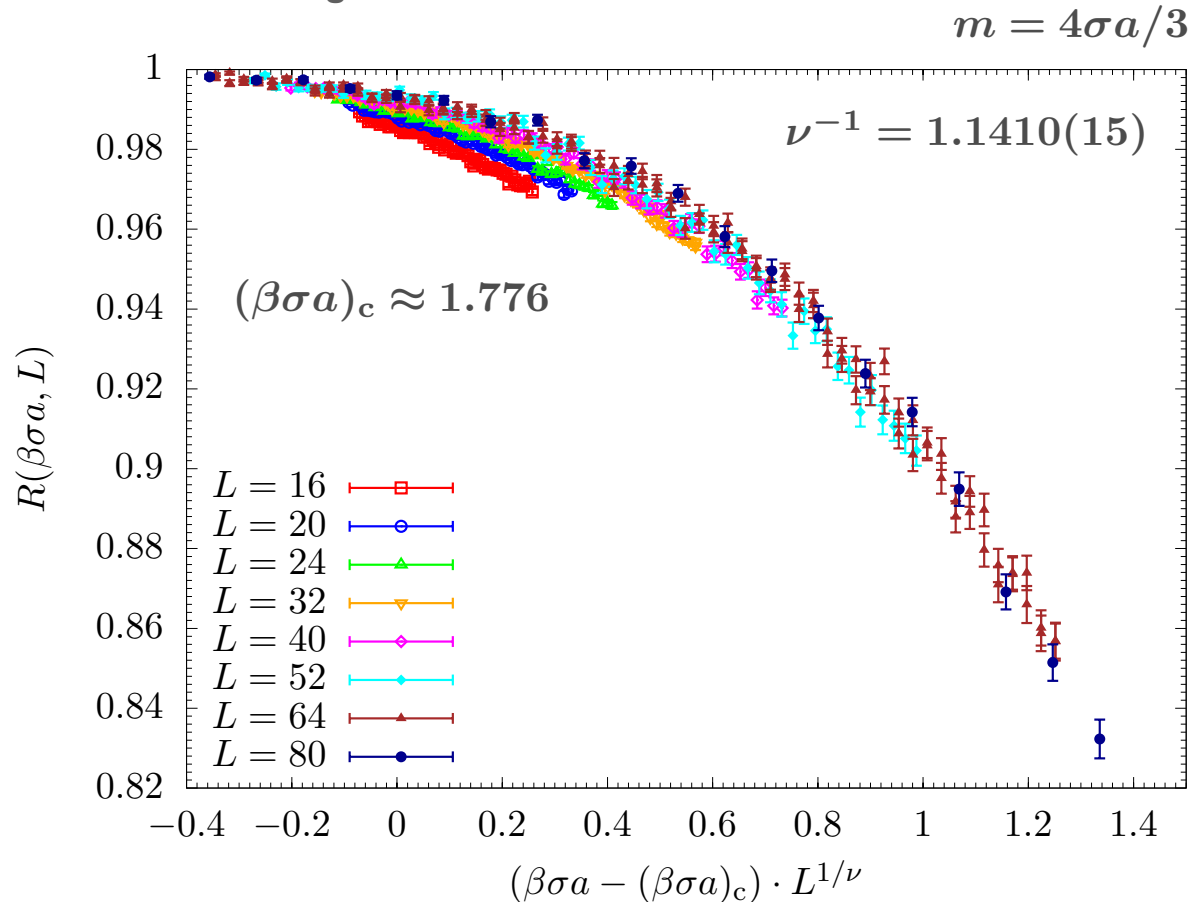
Percolation in Flux-Tube Model

- medium heavy quarks
still in Z_3 -Potts crossover region



Percolation in Flux-Tube Model

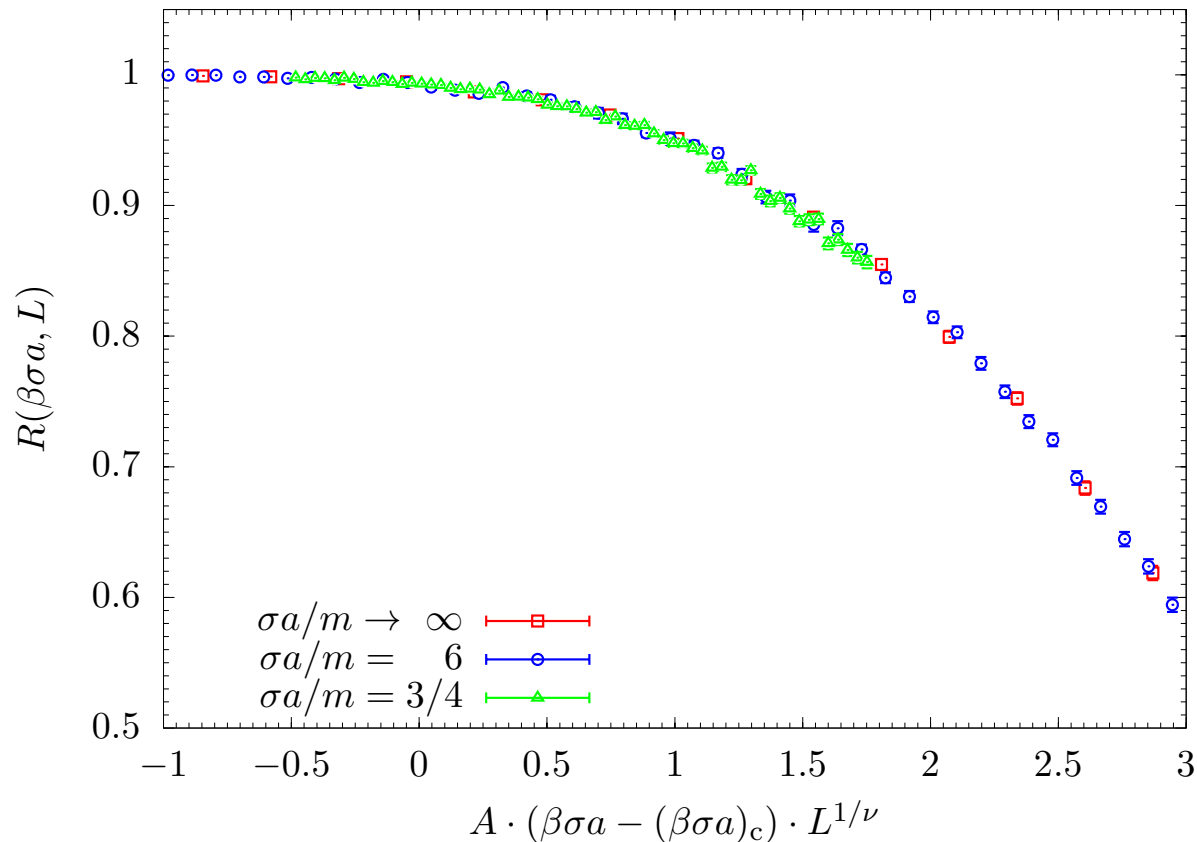
- medium heavy quarks
still in Z_3 -Potts crossover region



Percolation in Flux-Tube Model

- universal scaling function

combine $m = \{0, \sigma a/6, 4\sigma a/3\}$



Summary

- **Quarks and triality in a finite volume**

from FTs over stacks of closed center vortex sheets

- **Proof in two ways:**

[see Ghanbarpour & LvS, PRD 106 (2022) 054513]

1. **dualization of quark action**

Gattringer & Marchis, NPB 916 (2017) 627

Marchis & Gattringer, PRD 97 (2018) 034508

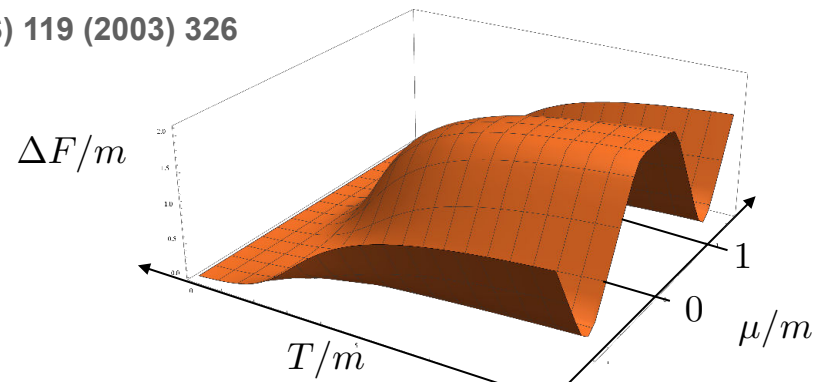
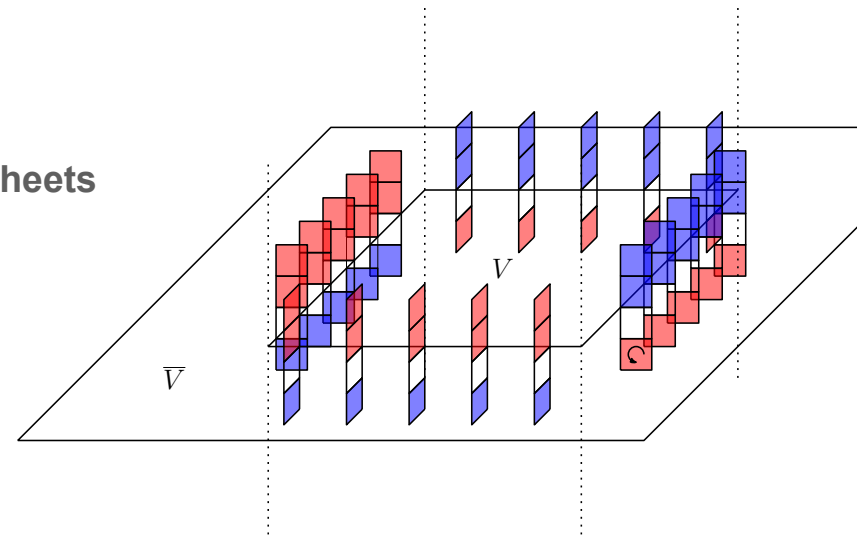
2. **transfer matrix approach**

Lüscher, Com. Math. Phys. 54 (1977) 283, Borgs & Seiler, Com. Math. Phys. 91 (1983) 329

Palumbo, NPB 645 (2002) 309, Mitrjushkin, NPB (PS) 119 (2003) 326

- **Illustration: heavy-dense QCD**

effective theory dual to flux-tube model

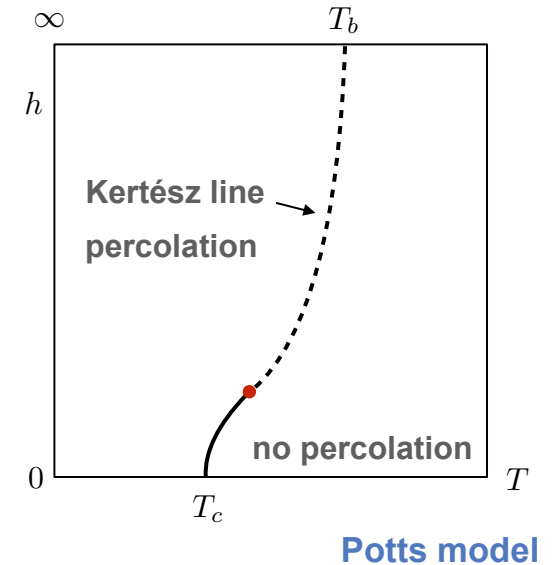
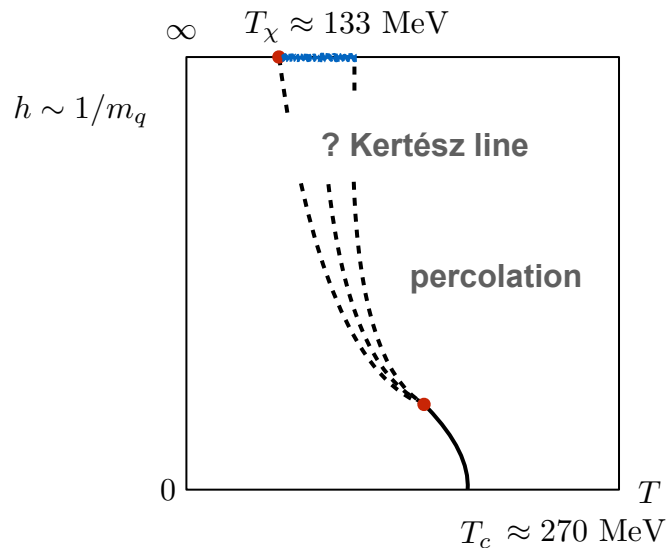


Summary & Outlook

- Percolation of electric fluxes in effective theory

geometric deconfinement phase transition
at strong coupling with static fermion determinant

- Percolation of electric fluxes in QCD



expect: geometric deconfinement phase transition

have: gauge invariant definition of fluxes
and spanning probability

Thank you for your attention!